Is there a cosmological Casimir effect?

On the reality of vacuum energy

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The Casimir effect

Neutral conducting plates attract each other. Why?

1. This is the long-range limit of the van der Waals force. Polarization fluctuations of the electrons in one plate induce such fluctuations in the other plate, causing a net attraction. Relativistic retardation (finite speed of light) must be taken into account in a quantitative treatment.

   But this is the hard way to do the calculation.

2. Because the modes of the EM field between the plates are discrete ($\omega_n = n\pi/L$ for separation L), the energy density in the EM field differs from that in free space:

$$\infty - \infty = -\frac{\pi^2}{720}\,\frac{1}{L^3}\,.$$

$$\text{So}\quad P = -\frac{\partial E}{\partial L} = -\frac{\pi^2}{200}\,\frac{1}{L^4}\,.$$

These are not **alternative** explanations. They are two ways of saying the same thing!

The vacuum fluctuations of the EM field mediate the **retarded** interaction of the vacuum fluctuations of the electrons in the plates.

I argue that if the force exists, then the vacuum energy must exist, else energy would not be conserved. It is the macroscopic potential energy of the configuration.

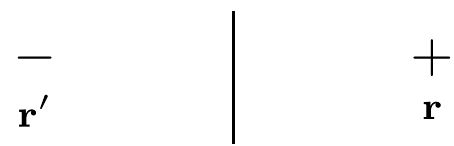
The method of images

As an alternative to eigenmodes, the energy density and pressure can be calculated from a Green function.

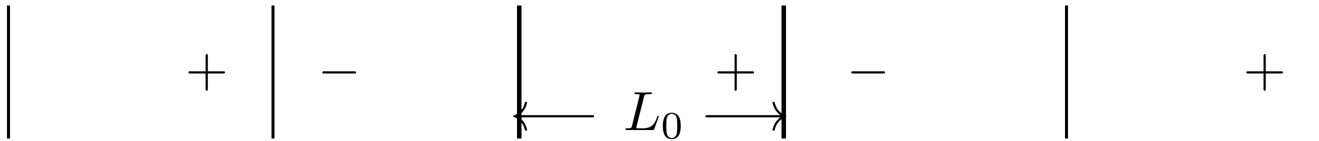
Free space: $G_0(\mathbf{r}, t; \mathbf{r}', t') = \dfrac{-1/2\pi^2}{(t-t')^2 + \|\mathbf{r}-\mathbf{r}'\|^2}\,.$

Plane boundary: (Suppress y, z.)

$$G(x,t;x',t') = G_0(x,t;x',t') - G_0(x,t;-x',t').$$



Slab: (Suppress t.) $G(x,x') = G_0(x,x') - G_0(x,-x') - G_0(x,2L_0-x') + G_0(x,2L_0+x') + G_0(x,-2L_0+x') + \cdots$.



The image set is periodic with period $L_1 = 2L_0$.

Periodic universe: $G(x,x') = G_0(x,x')$

$$+ G_0(x,2L_0+x') + G_0(x,-2L_0+x') + \cdots .$$

Conclusion: $G \neq G_0 \;\Rightarrow\;$ *A periodic space contains homogeneous but nonzero Casimir energy, which should affect the cosmological expansion.*

Casimir effect in cosmology?

Similar mathematics applies to every quantized field in a finite universe. There is a net vacuum energy density that depends on the size of the space (and the other geometry, and the field content of the model). This energy (and pressure) must appear in the Einstein equation — and cause a small contractive effect in a small Friedmann universe.

But the polarizable atoms and electrons have completely disappeared from this picture!

So, is this cosmological Casimir effect real, or a mathematical delusion? (Clearly it is **not** a van der Waals effect.)

It is sometimes argued that the Casimir energy is merely a bookkeeping device for calculating the van der Waals force between the conductors; in cosmology there are no conductors, hence there can't be a force.

Or, at least, there is no evidence for such a force, or for "real" vacuum energy.

“[T]he experimental confirmation of the Casimir effect does not establish the reality of zero-point fluctuations. ... [T]he concept of zero-point fluctuations is a heuristic and calculational aid in the description of the Casimir effect, but not a necessity. ... [T]here is no experimental evidence for the reality of zero-point energies in quantum field theory (without gravity).”

R. L. Jaffe, *Phys. Rev. D* **72**, 021301 (2005)

I do not completely disagree, especially with the last sentence. Ultimately physics is an experimental science. Only empirical evidence can settle definitively whether a cosmological Casimir effect exists.

But if it does not, then there is something wrong with our understanding of quantum field theory in general. In other words, I argue that Cosmological Casimir Energy Denial requires an illogical, inconsistent attitude toward the formalism of quantum field theory.

Disclaimer

I am **not** talking (today) about the cosmological constant.

- It is sometimes argued (sometimes by me) that the ubiquitous zero-point energy of all quantum fields amounts to a renormalization of the cosmological constant.
- It has been proposed (Milton, Kantowski, and Kao; Levin and Greene; Elizalde) that the cosmological constant term is induced from (renormalized) Casimir energy in extra dimensions.

The cosmological term in $T_{\mu\nu}$ is proportional to $g_{\mu\nu} = \operatorname{diag}(-1,1,1,1)$ in a local orthonormal frame. I am talking about an ordinary Casimir term in the macroscopic dimensions (**after** renormalization); it must be traceless (for a massless field); typically proportional to $\operatorname{diag}\left(1,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ or $\operatorname{diag}(-1,1,1,-3)$.

The example

Consider a massless scalar field with Dirichlet boundary conditions. The simplest case for calculations has curvature coupling constant $\xi = \frac{1}{4}$.

The publication

J. Phys. A: Math. Theor. 45 (2012) 374018 (24pp)
doi:10.1088/1751-8113/45/37/374018

Wedges, cones, cosmic strings and their vacuum energy

S A Fulling, C S Trendafilova, P N Truong, and J Wagner

> My junior colleagues are not responsible for any polemical or philosophical remarks by me that anyone may find objectionable.

Director's cut:

arXiv:1205.1818 [pdf, other]

Wedges, Cones, Cosmic Strings, and the Reality of Vacuum Energy

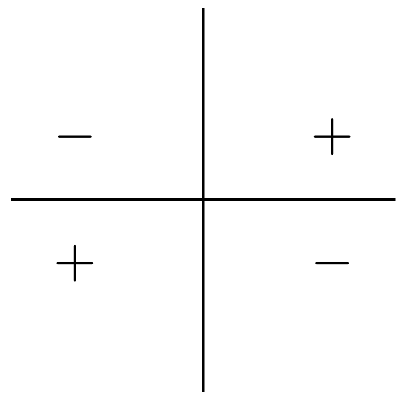
S. A. Fulling, C. S. Trendafilova, P. N. Truong, J. Wagner

Comments: 28 pages, 16 figures.
Special volume in honor of J. S. Dowker

Journal-ref: J. Phys. A: Math. Theor. 45 (2012) 374018 (minor revisions including title)

Wedges and cones

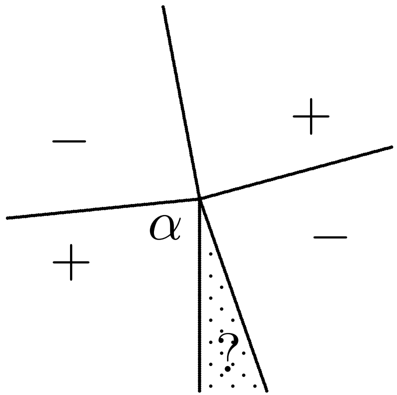
WEDGES WITH NICE ANGLES

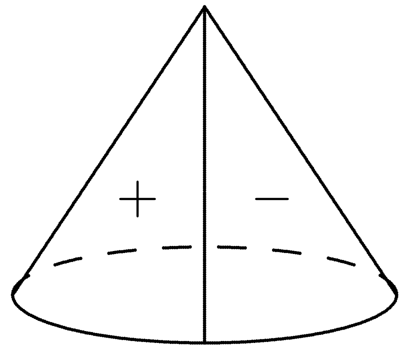


$N = 2$

If $\alpha = \frac{2\pi}{2N}$, the cylinder kernel (or other Green function) can be found by the classic method of images.

Wedges with bad (i.e., generic) angles





$2N$ copies of the wedge do not fit into the plane, but do fit into a cone of defect angle $2\pi - 2N\alpha$ (which may be negative: $0 < \alpha < \infty$).

Cones (infinitely thin cosmic strings)

So, what is the Green function on a cone? It must be periodic with period $\theta_1 = 2\pi -$ defect angle $= 2N\alpha$. There is no reason not to choose $N = 1$, so $\theta_1 = 2\alpha$ in the polar problems just as $L_1 = 2L_0$ in the Cartesian problems.

The analog of the free Green function G_0 is the one for the cone with angle $\theta_1 = \infty$, an infinite-sheeted Riemann surface. (z, t suppressed.)

$$G_{\theta_1}(\theta, \theta') = G_\infty(\theta, \theta') + G_\infty(\theta, \theta' + \theta_1) + \cdots .$$

Note that $G_{2\pi}$ is now the original free-space Green function in polar coordinates.

This infinite-sheeted cone is the *Dowker manifold*:

J. S. Dowker, *J. Phys. A* **10** (1977) 115, and later papers (channeling Sommerfeld and Carslaw on diffraction theory)

Historically, cone manifolds (Riemann surfaces) were introduced to study wedges by the method of images. More recently cosmic strings were studied by analogy with wedges, though the former are more elementary.

If we can calculate vacuum energy around cosmic strings, we can do it around wedges (though the understanding of divergences is more problematical in the latter case).

Vacuum energy around a cosmic string renormalizes the mass/length of the string. But there is no string at the vertex of a conducting wedge.

Polar–Cartesian comparison

In the polar plane,

$$G_{2\pi}(\theta, \theta') = \sum_{n=-\infty}^{\infty} G_\infty(\theta, \theta' + 2\pi n).$$

In the periodic universe,

$$G_{L_1}(x, x') = \sum_{n=-\infty}^{\infty} G_0(x, x' + nL_1).$$

$G_{2\pi}$ and G_0 are the free-space Green function.

In terms of normal modes, $G_{2\pi}$ is a sum over angular momentum quantum number, G_∞ an integral over it. G_{L_1} is a Fourier sum, G_0 a Fourier integral.

$G_{2\pi}$ and G_0 are the free-space Green function that gives the zero-point energy density that must be subtracted from that of any other configuration.

If you believe in the stress tensor of quantum field theory, all this is totally consistent and unsurprising.

But if you don't, you are forced into an untenable position:

- To calculate vacuum energy in the periodic universe, you must ignore the (mathematically appropriate) Fourier sum in favor of the integral (since you don't believe vacuum energy can exist in the absence of van der Waals sources).
- In polar coordinates, you must use the sum to get the right answer for empty Euclidean space. The integral gives something else, the energy density of the Dowker manifold.

I see no possibility of a theoretical justification for this ad hoc switch of point of view.

Green function and vacuum energy on cones

PARTIAL BIBLIOGRAPHY (besides Dowker)

- W. Lukosz, *Z. Phys.* **262** (1973) 327.
- A. G. Smith, in *The Formation and Evolution of Cosmic Strings*, ed. G. Gibbons et al. (C.U.P., 1990) 263.
- M. E. X. Guimarães and B. Linet, *Commun. Math. Phys.* **165** (1994) 297.

u, THE KEY INGREDIENT

Recall that Green functions in their full glory depend on variables $(r, \theta, z, t; r', \theta', z', t')$ (where t could also be x^0, ω, or s).

We can set t', z', $\theta' = 0$. Define u by

$$2rr' \cosh u = r^2 + r'^2 + z^2 + t^2$$

or

$$u = -\ln \frac{r_2 - r_1}{r_2 + r_1} \quad \text{where}$$

$$r_1 = \sqrt{(r - r')^2 + z^2 + t^2}, \quad r_2 = \sqrt{(r + r')^2 + z^2 + t^2}\,.$$

The cylinder kernel

$$G_\infty = -\frac{1}{2\pi^2 rr' \sinh u}\,\frac{u}{u^2+\theta^2}\,.$$

$$G_{\theta_1} = -\frac{1}{2\pi\theta_1 rr' \sinh u}\,\frac{\sinh\left(\frac{2\pi u}{\theta_1}\right)}{\cosh\left(\frac{2\pi u}{\theta_1}\right) - \cos\left(\frac{2\pi\theta}{\theta_1}\right)}\,.$$

$$G_{2\pi} = -\frac{1}{4\pi^2 rr'}\,\frac{1}{\cosh u - \cos\theta} = \frac{-1/2\pi^2}{t^2 + \|\mathbf{r}-\mathbf{r}'\|^2}\,.$$

Note that $G_\infty(\theta, \ldots)$ looks suspiciously like $G_{2\pi}(x - x', \ldots)$.

Correspondingly, G_{θ_1} has a structure similar to the G_{L_1} for the periodic universe.

In summary, G_∞ is found by separation of variables (sum over modes). G_{θ_1} can be found likewise, but also can be found from G_∞ by images (sum over paths). (G_∞ tells how to diffract a path off the isolated conical singularity.) These remarks extend to the energy density, and to wedges.

The energy density

With curvature coupling $\xi = \frac{1}{4}$, we need only t derivatives of G, with $r = r'$, $z = 0$, and $\theta - \theta' = 0$ ("on diagonal"). (Cutoff t taken small but nonzero.)

$$T_{00}(r,t) = -\frac{1}{2}\frac{\partial^2 G}{\partial t^2} = \frac{1}{t^4}\, F\left(\frac{r}{t}\right).$$

We have formulas, but for most cases they are complicated and uninformative. *Mathematica* plotting proves indispensable. (Dashed lines will be for no cutoff.)

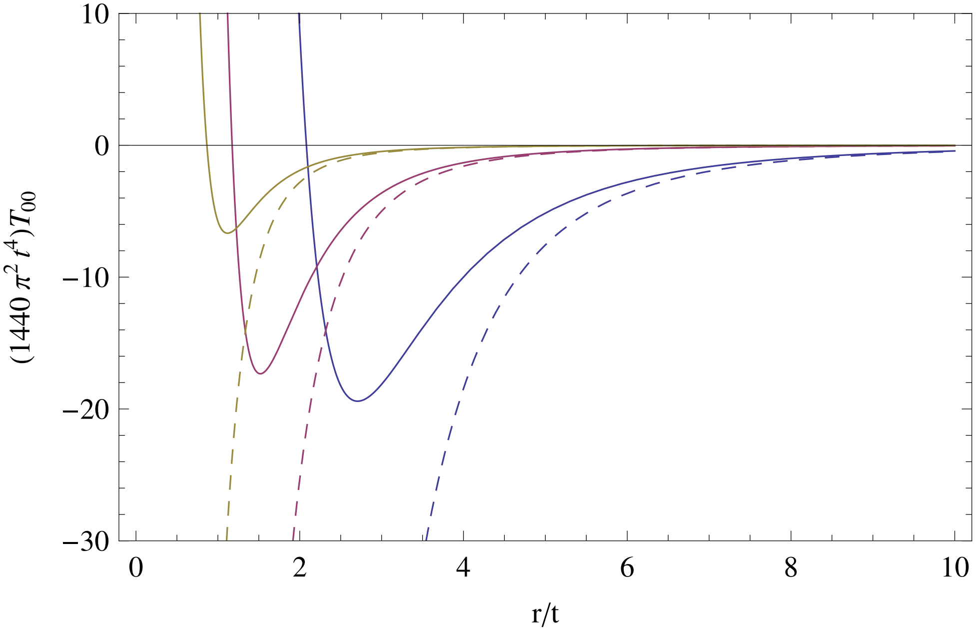
Exception: **conformal coupling** ($\xi = \frac{1}{6}$). In the limit of no cutoff, the energy density is independent of the angle coordinate:

$$T_{00} = \frac{1}{1440 r^4 \alpha^2} \left(\frac{\pi^2}{\alpha^2} - \frac{\alpha^2}{\pi^2} \right)$$

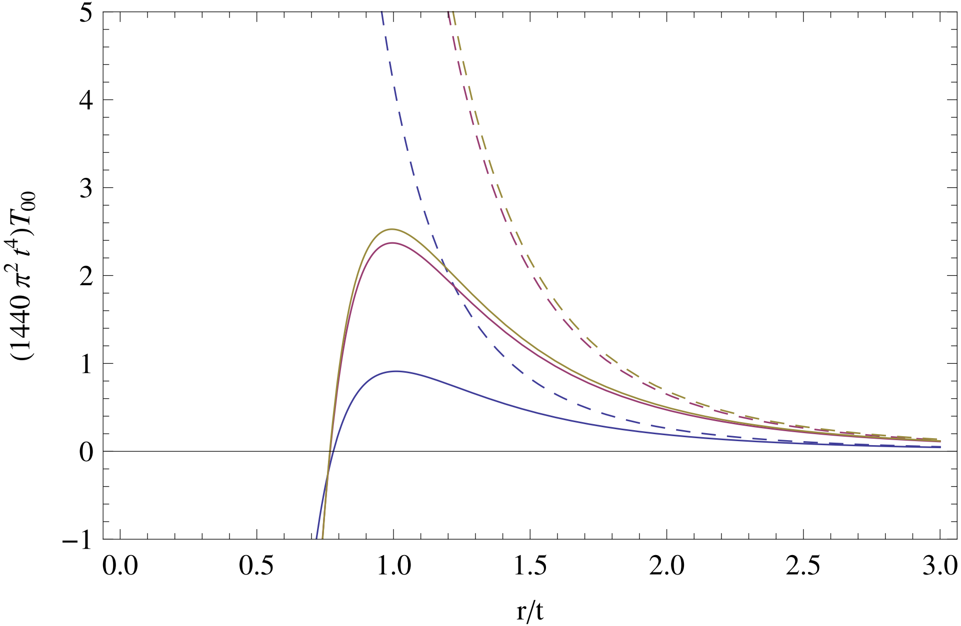
for a wedge of angle α. (D. Deutsch and P. Candelas, *Phys. Rev. D* **20** (1979) 3063) There is a divergence at the tip only, not the wedge boundary.

There is a **torque anomaly** (energy and pressure formulas don't agree), but that is for another talk.

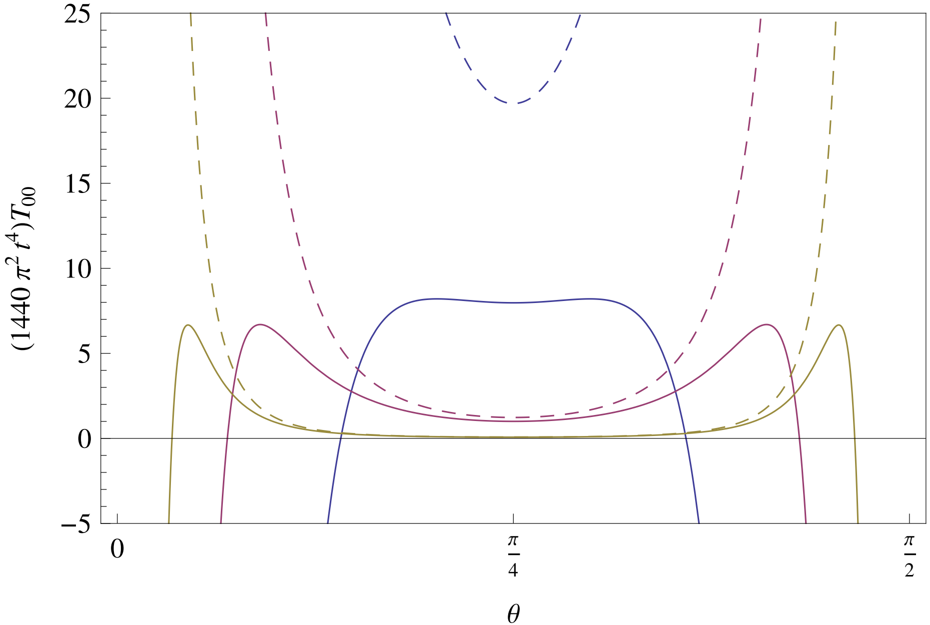
Energy density ($\xi = \frac{1}{4}$) for cone angles $\frac{\pi}{4}$, $\frac{3\pi}{5}$, π



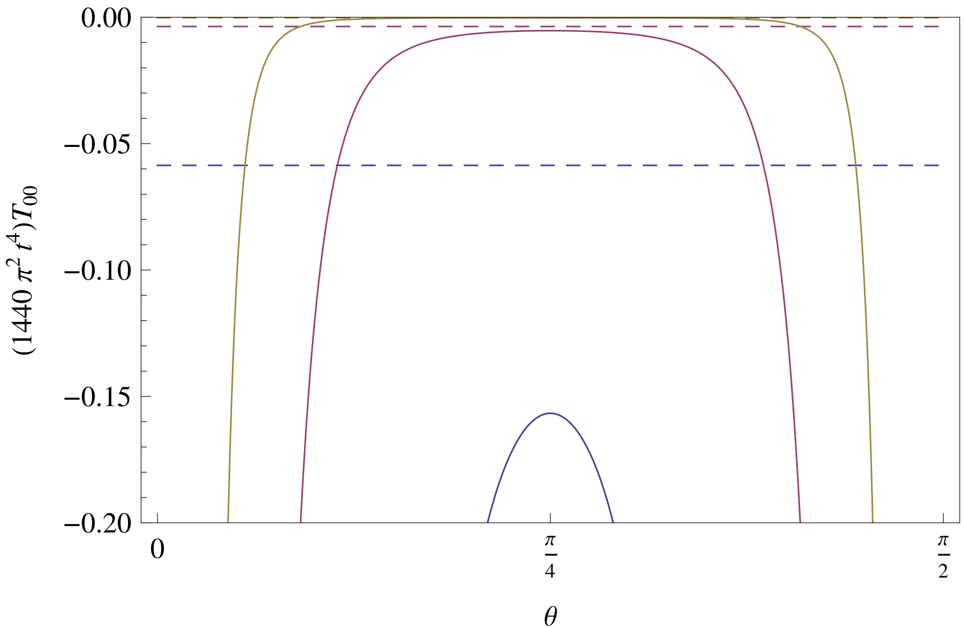
ENERGY DENSITY ($\xi = \frac{1}{4}$) FOR CONE ANGLES $\frac{5\pi}{2}$, 8π, ∞



WEDGE ENERGY DENSITY ($\xi = \frac{1}{4}$) FOR $r = 2,\ 4,\ 8$



Wedge energy density (conformal) for $r = 4,\ 8,\ 16$

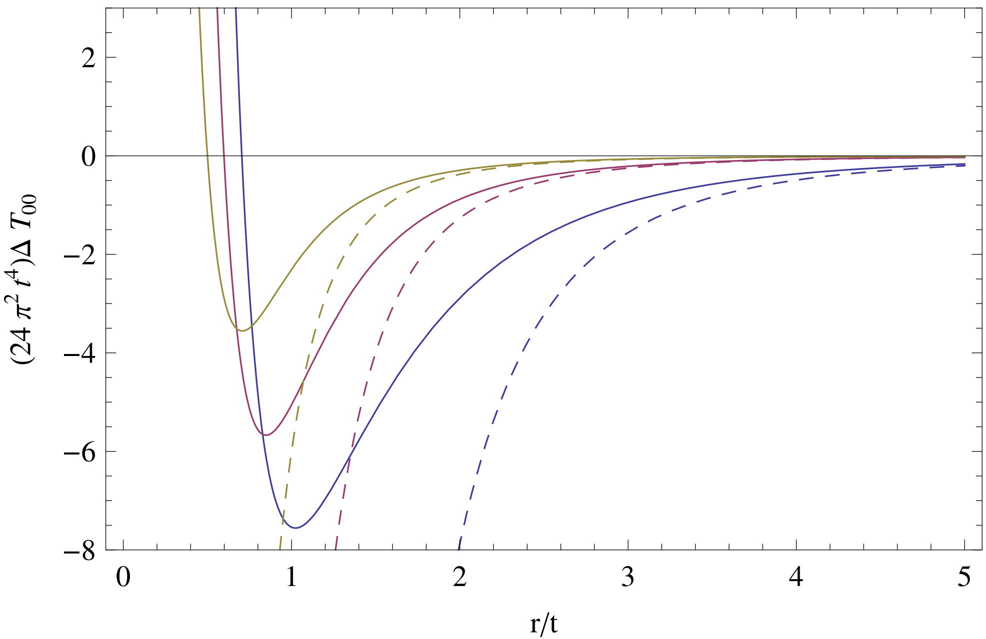


Summary

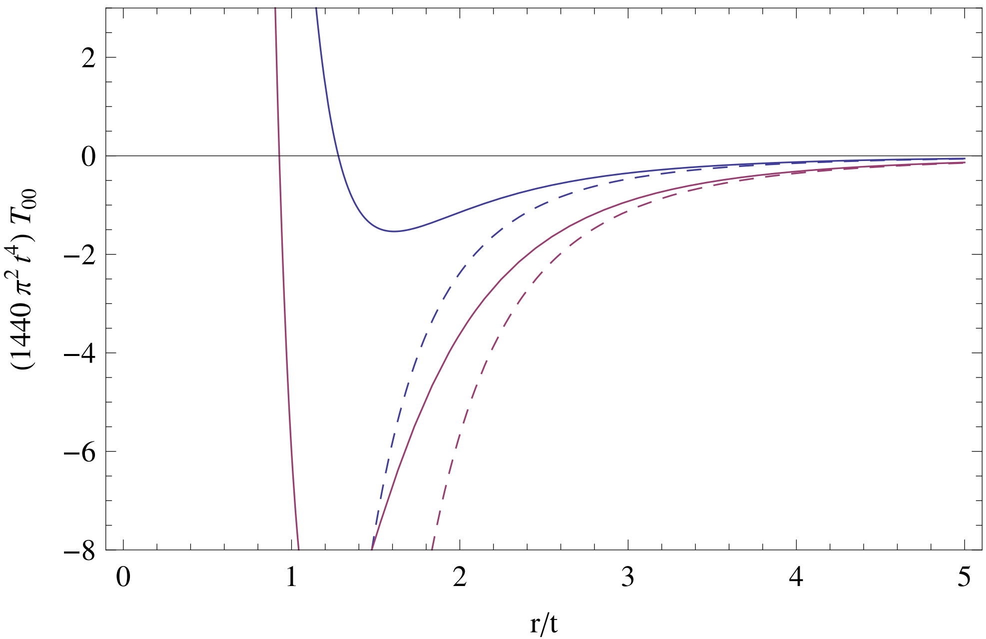
1. Wedge $\Leftarrow$ cone + image(s).
2. Cone $\Leftarrow$ Dowker + periodic.
3. Flat space = cone of angle 2π. Discreteness of angular modes cancels vacuum energy of Dowker space.
4. We calculate **local energy density** for **all values** of the curvature coupling constant ξ.

5. Conformal coupling ($\xi = \frac{1}{6}$) removes plane and wedge-side divergences
(⇒ flat function of angle in limit of no cutoff)
but not cone and wedge-vertex divergences.
6. Total energy (per length) is independent of ξ **only** when cutoff is retained. That is, although the ξ-dependent term in the density is a total divergence, there is nonuniform convergence near the boundary as the cutoff is removed. (DeWitt; Ford & Svaiter)

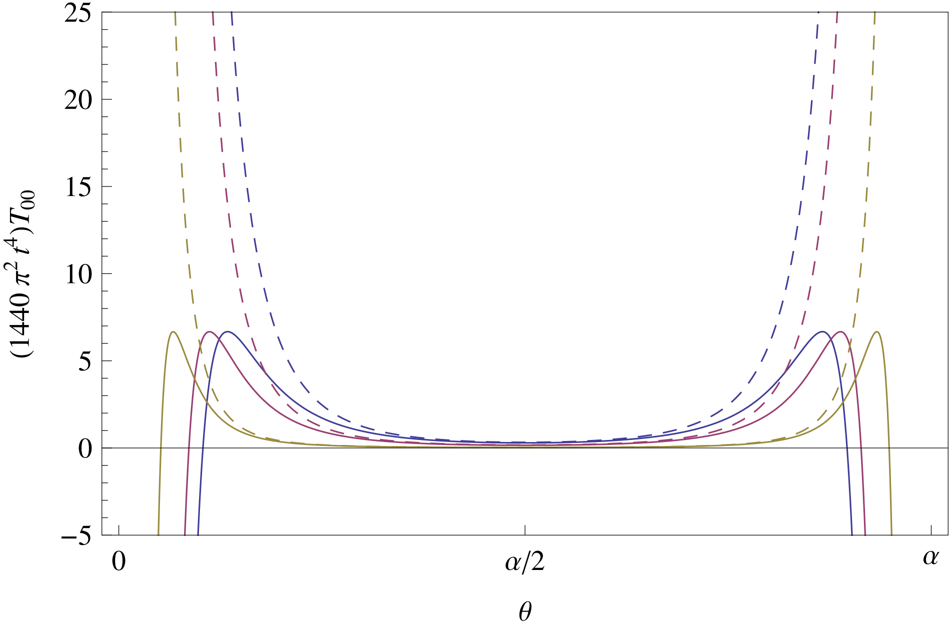
Curvature coupling term for cone angles $\frac{\pi}{4}$, $\frac{3\pi}{5}$, π



ENERGY DENSITY FOR CONE ANGLE $\frac{4\pi}{5}$ AND $\xi = \frac{1}{4},\ \frac{1}{6}$



WEDGE ANGLES $\frac{\pi}{3}$, $\frac{2\pi}{5}$, $\frac{2\pi}{3}$; $r = 8$, $\xi = \frac{1}{4}$



WEDGE ANGLES $\frac{\pi}{3}$, $\frac{2\pi}{5}$, $\frac{2\pi}{3}$; $r = 8$, $\xi = \frac{1}{6}$