

The Kerr Solution

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The Kerr Metric

$$ds^2 = ds_0^2 + 2mr/(r^2 + a^2 \cos^2 \theta)k^2, \quad k = -d(t + r) + a \sin \theta d\phi$$
$$ds_0^2 = (r^2 + a^2 \cos^2 \theta)(d\theta^2 + \sin^2 \theta d\phi^2) + (dt - dr + a \sin^2 \theta d\phi)k$$

$$(r + ia)e^{i\phi} \sin \theta = x + iy, \quad r \cos \theta = z,$$

gives the **Kerr-Schild form** of the metric,

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2mr^3}{r^4 + a^2 z^2} \left[dt + \frac{z}{r} dz \right. \\ \left. + \frac{r}{r^2 + a^2} (x dx + y dy) - \frac{a}{r^2 + a^2} (x dy - y dx) \right]^2. \quad (1)$$

where the surfaces of constant r are confocal ellipsoids of revolution,

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.$$

The Angular Momentum of Central Body

The metric was expanded in powers of R^{-1} , where $R = x^2 + y^2 + z^2$ is the usual Euclidean distance from the origin, the center of the source,

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{R}(dt + dR)^2 - \frac{4ma}{R^3}(xdy - ydx)(dt + dR) + O(R^{-3})$$

If $x^\mu \rightarrow x^\mu + a^\mu$ is an infinitesimal transformation, then $ds^2 \rightarrow ds^2 + 2da_\mu dx^\mu$,

$$a_\mu dx^\mu = -\frac{ma}{R^2}(xdy - ydx) \Rightarrow 2da_\mu dx^\mu = -\frac{4ma}{R^3}(xdy - ydx)dR,$$

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{R}(dt + dR)^2 - \frac{4ma}{R^3}(xdy - ydx)dt + O(R^{-3}).$$

It is rotating! The mass = m , Angular Momentum = ma .

Boyer-Lindquist Coordinates

All cross terms between $\{dr, d\theta\}$ and $\{dt, d\phi\}$ can be eliminated

$$dt' = dt + A dr + B d\theta, \quad d\phi' = d\phi + C dr + D d\theta.$$

where the coefficients can be found algebraically.

$$dt \rightarrow dt + \frac{2mr}{\Delta} dr \quad d\phi \rightarrow -d\phi + \frac{a}{\Delta} dr, \quad \Delta = r^2 - 2mr + a^2.$$

The right hand sides of the first two equations are clearly perfect differentials. This **Boyer-Lindquist form** is the most widely used form of the metric,

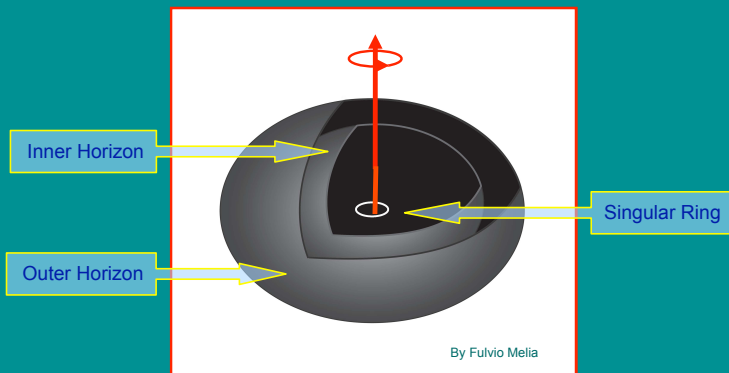
$$ds^2 = \frac{\Theta}{\Delta} dr^2 + \Theta d\theta^2 - \frac{\Delta}{\Theta} [dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\Theta} [(r^2 + a^2) d\phi - a dt]^2,$$

$$\text{where} \quad \Theta = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2mr + a^2$$

The two event horizons are the surfaces $r = r_{\pm}$ where r_{\pm} are the roots of $\Delta = 0$,

$$\Delta = r^2 - 2mr + a^2 = (r - r_+)(r - r_-).$$

Event Horizons for Kerr Black Hole



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