The Kerr Solution

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The Kerr Metric

$$ds^2 = ds_0^2 + 2mr/(r^2 + a^2\cos^2\theta)k^2, \qquad k = -d(t+r) + a\sin\theta d\phi$$
$$ds_0^2 = (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2) + (dt - dr + a\sin^2\theta d\phi)k$$

$$(r + ia)e^{i\phi}\sin\theta = x + iy, \quad r\cos\theta = z,$$

gives the Kerr-Schild form of the metric,

$$\begin{aligned} ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2mr^3}{r^4 + a^2z^2}[dt + \frac{z}{r}dz \\ + \frac{r}{r^2 + a^2}(xdx + ydy) - \frac{a}{r^2 + a^2}(xdx - ydy)]^2. \end{aligned} \tag{1}$$

where the surfaces of constant r are confocal ellipsoids of revolution,

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.$$

The Angular Momentum of Central Body

The metric was expanded in powers of R^{-1}, where $R = x^2 + y^2 + z^2$ is the usual Euclidean distance from the origin, the center of the source,

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{R}(dt + dR)^2$$
$$- \frac{4ma}{R^3}(xdy - ydx)(dt + dR) + O(R^{-3})$$

If $x^\mu \to x^\mu + a^\mu$ is an infinitesimal transformation, then $ds^2 \to ds^2 + 2da_\mu dx^\mu$,

$$a_\mu dx^\mu = -\frac{ma}{R^2}(xdy - ydx) \quad \Rightarrow \quad 2da_\mu dx^\mu = -\frac{4ma}{R^3}(xdy - ydx)dR,$$

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{R}(dt + dR)^2$$
$$- \frac{4ma}{R^3}(xdy - ydx)dt + O(R^{-3}).$$

It is rotating! The mass $= m$, Angular Momentum $= ma$.

Boyer-Lindquist Coordinates

All cross terms between $\{dr, d\theta\}$ and $\{dt, d\phi\}$ can be eliminated

$$dt' = dt + Adr + Bd\theta, \quad d\phi' = d\phi + Cdr + Dd\theta.$$

where the coefficients can be found algebraically.

$$dt \to dt + \frac{2mr}{\Delta}dr \qquad d\phi \to -d\phi + \frac{a}{\Delta}dr, \qquad \Delta = r^2 - 2mr + a^2.$$

The right hand sides of the first two equations are clearly perfect differentials. This Boyer-Lindquist form is the most widely used form of the metric,

$$ds^2 = \frac{\Theta}{\Delta}dr^2 + \Theta d\theta^2 - \frac{\Delta}{\Theta}[dt - a\sin^2\theta d\phi]^2 + \frac{\sin^2\theta}{\Theta}[(r^2 + a^2)d\phi - adt]^2,$$

$$\text{where} \qquad \Theta = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2mr + a^2$$

The two event horizons are the surfaces $r = r_\pm$ where $r_\pm$ are the roots of $\Delta = 0$,

$$\Delta = r^2 - 2mr + a^2 = (r - r_+)(r - r_-).$$

Fulvio's Diagram of Event Horizons