Global Coordinates for Schwarzschild
or
Lost opportunities

Over the past few years I have become interested in the development of the understanding of the horizon.

Many of you will know much of what I am going to say, but I hope there will be some new things for at least some.

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UBC-Sep 2017

Schwarzschild 1916 (while at the Eastern Front WWI)

$$ds^2 = (1 - \frac{2M}{r})dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2(d\Omega^2)$$

.

$$d\Omega^2 = d\theta^2 + sin(\theta)^2 d\phi^2$$

(actually not this since he used coordinates so that |g|=1)

(Schwarzschild singularity r=2M. Total confusion)

Lancsos recognized that singularities could come from coordinates, but lesson did not sink in

Assume 2M=1 for simplicity.

1921 both Gullstrand and Painleve found metric “regular” at r=1

$$ds^2 = (1 - \frac{1}{r})dt^2 \pm 2\sqrt{\frac{1}{r}}dtdr - dr^2 - r^2(d\Omega^2)$$

but argued that this proved that Einstein’s theory was wrong as it had too many solutions for the same physics (time independent spherical symmetry) .

Einstein argued that this was just an aspect of coordinate freedom, but seems not to have actually shown it. He seems to have dismissed this metric (which is completely regular at r=1) because he did not like the off diagonal terms and . did not notice it made r=1 regular.

1922 Eddington showed that there is another solution (which I believe he recognized as a coordinate transform of Schw.

$$ds^2 = (1 - \frac{1}{r})d\tilde{t}^2 + \frac{2}{r}d\tilde{t}dr - (1 + \frac{1}{r})dr^2 + r^2 d\Omega^2$$

This is not what we call the Eddington- Finkelstein coord. which replaces dt-dr by the null coordinate du.
Neither of them wrote down the nul form of metric.
That was first done by Penrose who attributed them to first Finkelstein (1958)and then Eddinton and Finkelstein.

Eddington did not note that these are regular coord at r=1
He was concerned with the relation to Whitehead’s theory.

Finkelstein completely understood that his coordinates made a horizon regular and the horizon acted as a one-way surface

The above is also what is called the Kerr-Schild form of Schw.

Finally in 1933, Lemaitre not only proved that the PG system was just a coordinate transformation of Schw. but also introduced a new set of coords. in which

$$ds^2 = d\tau^2 - \frac{1}{r}d\sigma^2 + r^2 d\Omega^2$$

$$r = [\frac{3}{2}(\sigma + \tau)]^{\frac{2}{3}}$$

In the process of deriving these coordinates, he also went through the PG coordinates and showed they are just a coordinate change from Schw.

Note that these coord. look regular everywhere except r=0. But they have incomplete null geodesics at r=1, $\sigma = \infty$

He also emphasised that these coordinates remove the singularity at r=1, proving it is purely a coordinate artifact

Did not clear up the confusion.

1935 Einstein and Rosen (The bridge paper, which should really be called the Flamm (1916) bridge) wrote the “well known” metric for accelerated observer

$$ds^2 = \rho^2 d\tau^2 - d\rho^2 - dx^2 - dy^2$$

is a solution of Einstein eqn and has zero curvature, and they explicitly give the coord transformation which gives this metric from the flat spacetime coordinates. They recognize the similarity of this to Schw. and even put Schw into the form with $u^2 dt^2 - du^2$... but do not realise that this could remove the Schw singularity. They argue that r=1 in Schw is a matter sheet just as $\rho = 0$ in the accelerated metric above was a uniform matter sheet for them.

Einstein Rosen:

§1. A Special Kind of Singularity and Its Removal

The first step to the general theory of relativity was to be found in the so-called "Principle of Equivalence": If in a space free from gravitation a reference system is uniformly accelerated, the reference system can be treated as being "at rest," provided one interprets the condition of the space with respect to it as a homogeneous gravitational field. As is well known the latter is exactly described by the metric field[1]

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + \alpha^2 x_1^2 dx_4^2. \qquad (1)$$

The $g_{\mu\nu}$ of this field satisfy in general the equations

$$R^i_{klm} = 0, \qquad (2)$$

and hence the equations

$$R_{kl} = R^m_{klm} = 0. \qquad (3)$$

The $g_{\mu\nu}$ corresponding to (1) are regular for all finite points of space-time. Nevertheless one cannot assert that Eqs. (3) are satisfied by (1) for *all* finite values of $x_1, \cdots, x_4$. This is due to the fact that the determinant g of the $g_{\mu\nu}$ vanishes for $x_1 = 0$. The contravariant $g^{\mu\nu}$ therefore become infinite and the tensors R^i_{klm} and R_{kl} take on the form 0/0. From the standpoint of Eqs. (3) the hyperplane $x_1 = 0$ then represents a singularity of the field.

We now ask whether the field law of gravitation (and later on the field law of gravitation and electricity) could not be modified in a natural way without essential change so that the solution (1) would satisfy the field equations for all finite points, i.e., also for $x_1 = 0$. W. Mayer has called our attention to the fact that one can make R^i_{klm} and R_{kl} into rational functions of the $g_{\mu\nu}$ and their first two derivatives by multiplying them by suitable powers of g. It is easy to show

[1] It is worth pointing out that this metric field does not represent the whole Minkowski space but only part of it. Thus, the transformation that converts

$$ds^2 = -d\xi_1^2 - d\xi_2^2 - d\xi_3^2 + d\xi_4^2$$

into (1) is

$$\xi_1 = x_1 \cosh \alpha x_4, \qquad \xi_2 = x_2,$$
$$\xi_3 = x_3, \qquad \xi_4 = x_1 \sinh \alpha x_4.$$

It follows that only those points for which $\xi_1^2 \geq \xi_4^2$ correspond to points for which (1) is the metric.

Equivalence": If in a space free from gravitation a reference system is uniformly accelerated, the reference system can be treated as being "at rest," provided one interprets the condition of the space with respect to it as a homogeneous gravitational field. As is well known the latter is exactly described by the metric field[1]

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$$u^2 = r - 2m$$

$$ds^2 = -4(u^2+2m)du^2 - (u^2+2m)^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{u^2}{u^2+2m}dt^2. \qquad (5a)$$

Flamm 1916

Geometry of the t=const surface.

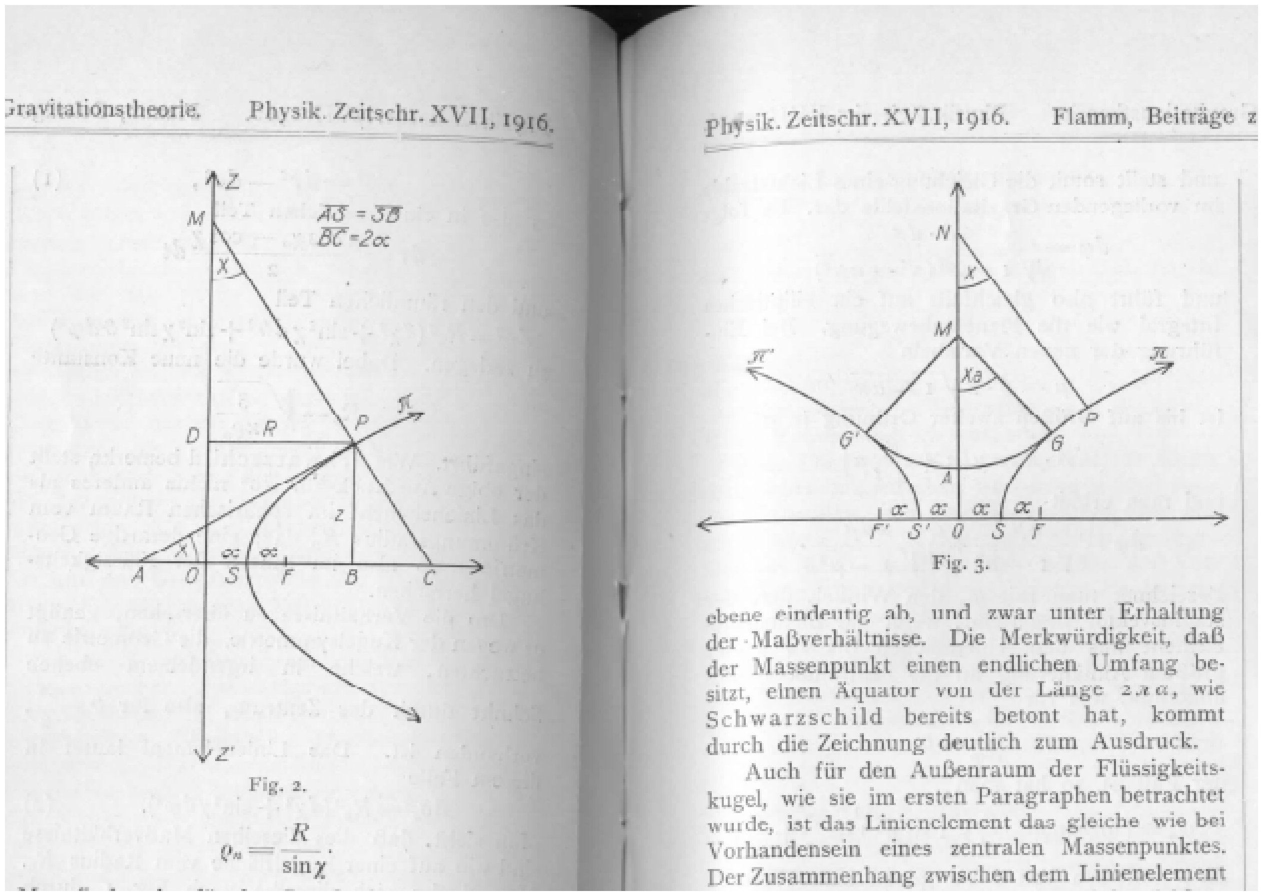

Gravitationstheorie. Physik. Zeitschr. XVII, 1916.

Fig. 2.

$$\varrho_n = \frac{R}{\sin\chi}.$$

Physik. Zeitschr. XVII, 1916. Flamm, Beiträge z

Fig. 3.

ebene eindeutig ab, und zwar unter Erhaltung der Maßverhältnisse. Die Merkwürdigkeit, daß der Massenpunkt einen endlichen Umfang besitzt, einen Äquator von der Länge $2\pi\alpha$, wie Schwarzschild bereits betont hat, kommt durch die Zeichnung deutlich zum Ausdruck.

Auch für den Außenraum der Flüssigkeitskugel, wie sie im ersten Paragraphen betrachtet wurde, ist das Linienelement das gleiche wie bei Vorhandensein eines zentralen Massenpunktes. Der Zusammenhang zwischen dem Linienelement

$$(1-\frac{2m}{r})dt^2 - \frac{dr^2}{(1-\frac{2M}{r})} - r^2 d\Omega^2$$

movie Interstellar: m<<M
metric completely regular everywhere.
m>M, true singularity
(only pressure/tension needed-- no energy)

Flamm shares with Einstein-Rosen-- mass at junction

There was another process going on.

1950-- Synge showed that there was a coordinate system which made everything regular. Covered all of Schw, and all geodesics either infinte or terminated at a singularity at r=0. (Actually he thought his coordinates all provided an analytic exptention through r=0, as well as r=1)

Far less known (completely unkown?) than the Kruscal extention, of which more later.

Go back to Einstein Rosen.

$$ds^2 = q d\tau^2 - \frac{dq^2}{q} - dx^2 - dy^2$$

$$ds^2 = \rho^2 d\tau^2 - d\rho^2 - dx^2 - dy^2$$

First to second by defining ρ to be proper distance to horizon.

$$ds^2 = (1 - \frac{1}{r})dt^2 - \frac{dr^2}{1 - \frac{1}{r}} - r^2(d\Omega^2)$$

$$R = \int_1 \frac{dr}{\sqrt{1 - 1/r}} = \sqrt{r-1}\left(\sqrt{r} + \frac{asinh(\sqrt{r-1})}{\sqrt{r-1}}\right)$$

$$1 - \frac{1}{r} = \frac{R^2}{r\left(\sqrt{r} + \frac{asinh(\sqrt{r-1})}{\sqrt{r-1}}\right)}$$

Define $$F(r) = \frac{4}{r\left(\sqrt{r} + \frac{asinh(\sqrt{r-1})}{\sqrt{r-1}}\right)^2}$$

$$T = Rsinh(t/2)$$

$$Q = Rcosh(t/2)$$

The metric becomes

$$ds^2 = F(r(R))R^2 d(\frac{t}{2})^2 - dR^2 - r(R)^2 d\Omega^2$$

$$= (F(r(R)) - 1)R^2 d\frac{t}{2}^2 + R^2 d\frac{t}{2}^2 - dR^2 - r(R)^2 d\Omega^2$$

$$ds^2 = (F(r(R)) - 1)(QdT - TdQ)^2 + dT^2 - dQ^2 - r(\sqrt{Q^2 - T^2})^2 d\Omega^2$$

The first term is 0 near the horizon. The others are just “flat” spacetime

- All of the terms are analytic for r>0 despite the apparent branch point at r=1. All terms are actually functions of r-1.
- This metric was the first global covering of the whole of the analytic extended Schw. geometry. And almost noone knows about it (?). It could easily have been discovered by Einstein and Rosen 18 years earlier.

Publication of Kruscal metric (1960)-- [from Charlie Misner]

It was discovered by Kruscal in the mid to later 50’s, but not written up. He mentioned it to Wheeler over lunch one day. Wheeler wrote it into his big book (Wheeler constantly carried abound a large ledger book into which he would write talks, conversations, etc. It would be great if we could ally with his repositories (Am. Phil Soc Philadephia, and Dolph Briscoe Center for Am. History in Austin to get at least digital copies here.) (Note that another live person who does the same is Don Page)

A while later Finkelstein sent Misner a copy of his paper on the horizon of Schw. and the one way membrane, which Misner showed to Wheeler. Wheeler remembered the Kruskal conversation. Kruscal had gone off on sabatical, so Wheeler wrote up the paper with Kruscal as author and sent if off to Phys Rev. Apparently, the first Kruscal knew he had written the paper was when he received the page or galley proofs.

Derivation: can follow a similar procedure to that of Synge: Write

$$ds^2 = G(\rho)(\frac{1}{4}\rho^2 dt^2 - d\rho^2) - r(\rho)^2 d\Omega^2$$

$$\frac{d\rho}{dr} = \frac{1}{2}\frac{\rho}{1-\frac{1}{r}}$$

$$ds^2 = \frac{e^{-r/2}}{r}\left(\rho^2 d(\frac{t}{2})^2 - d\rho^2\right) - r(\rho)^2 d\Omega^2$$

$$\tau = \rho sinh(\frac{t}{2})$$

$$\rho^2 = \tau^2 - \chi^2$$

$$\chi = \rho cosh(\frac{t}{2})$$

$$\rho^2 = (r-1)e^r$$

$$ds^2 = \frac{e^{-r/2}}{r}(d\tau^2 - d\chi^2) + r(\rho)^2 d\Omega^2$$

1965- Israel, 1971-Newman Pajerski

New global coordinate transformation which allowed the new coordinates and metric to be explicit (rather than implicit) function of coordinates.

$$
\begin{aligned}
ds^2 &= \frac{z^2}{r} dU^2 + 2dU\,dz - r^2 d\Omega^2 \\
r &= 1 + zU
\end{aligned}
$$

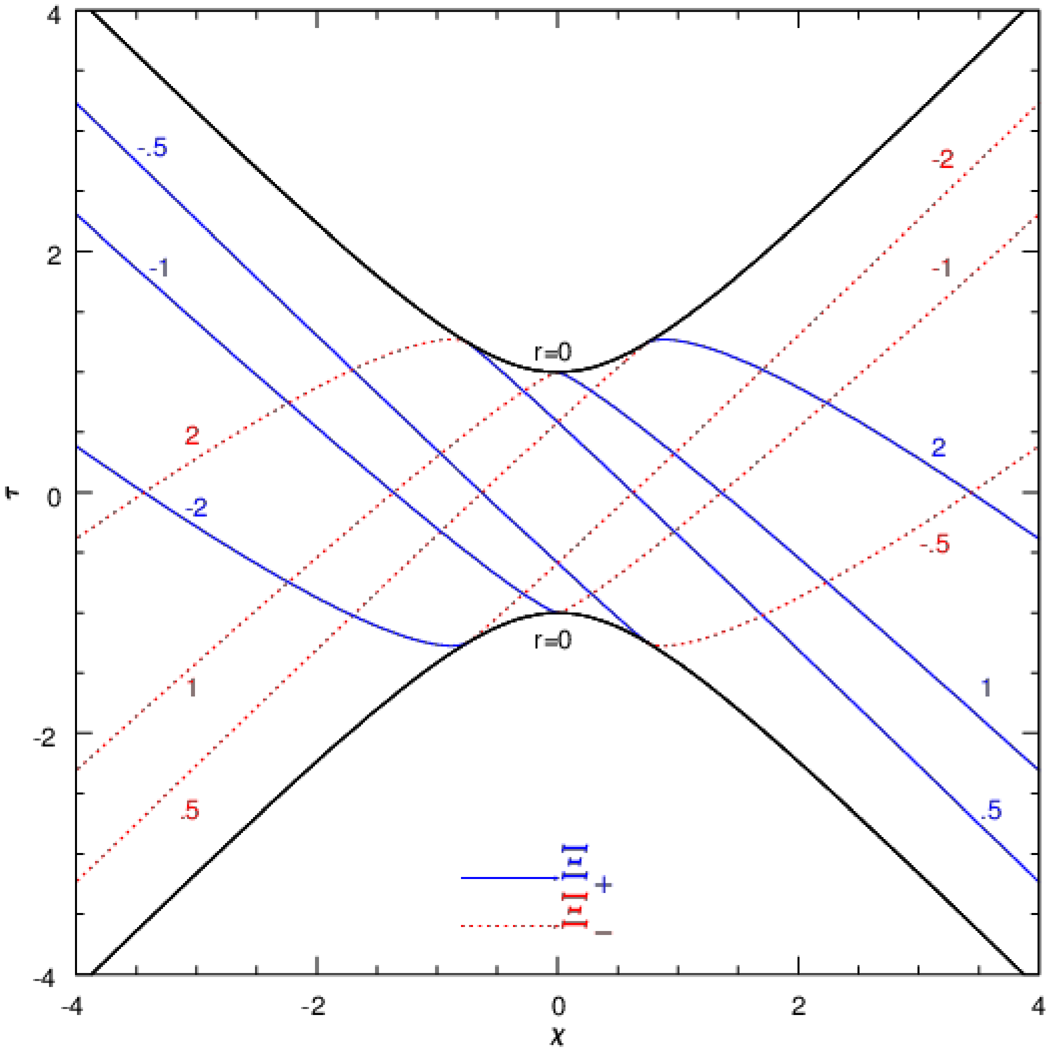
U const hypersurfaces are null (closely related to the Kruscal null coordinate), while z const are timelike (and are the affine parameter along the U=const)

These are also complete coordinates with cover the whole spacetime, and have the advantage of algebraic simplicity.

For both of the coordinates: Painleve Gulstrand (PG), and Lemaitre, one can choose the surfaces (flat constant time in the case of PG) and define a global set of coordinates which cover all of the Schw analytically extended spacetime

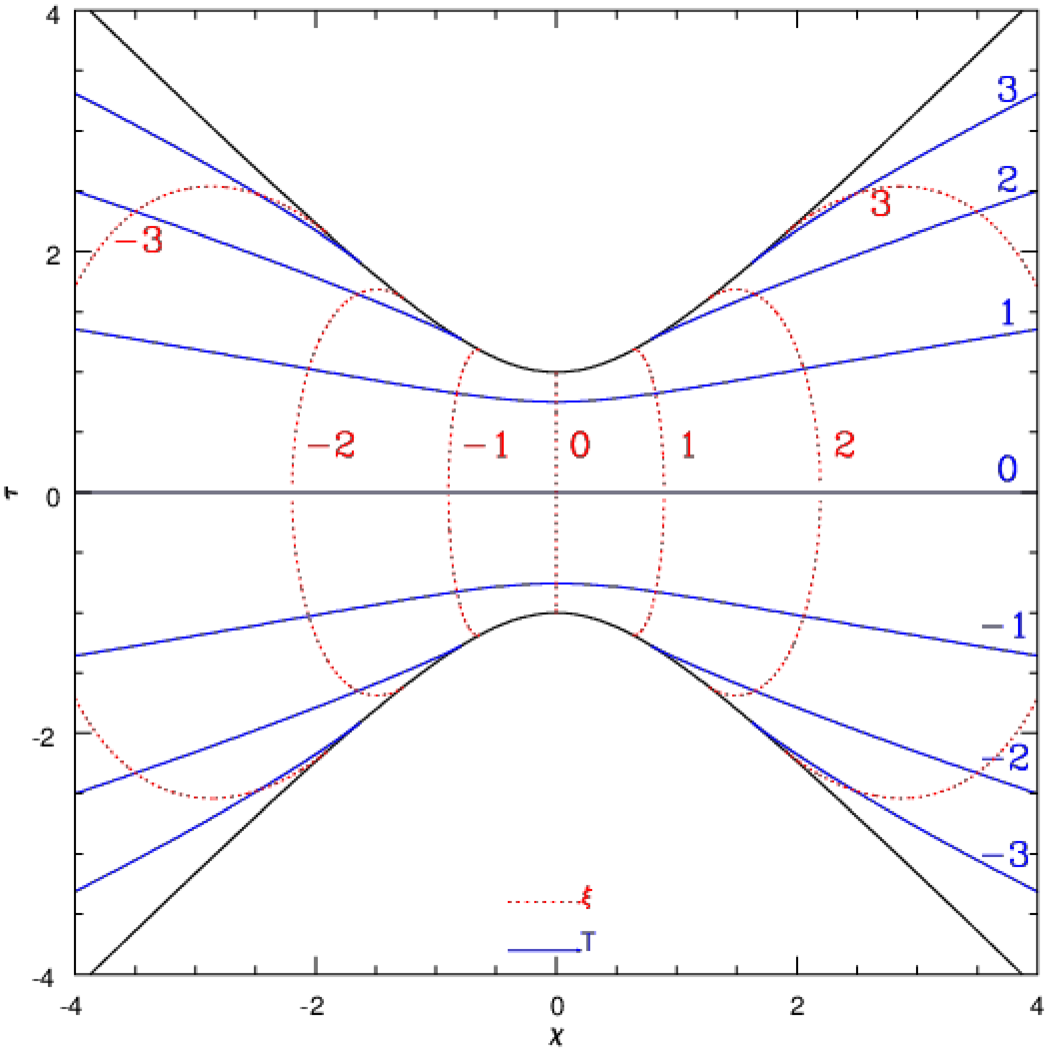
GRAPHS: In each case we graph the constant coordinate surfaces in the Kruscal set of coordinates.

PG coordinates

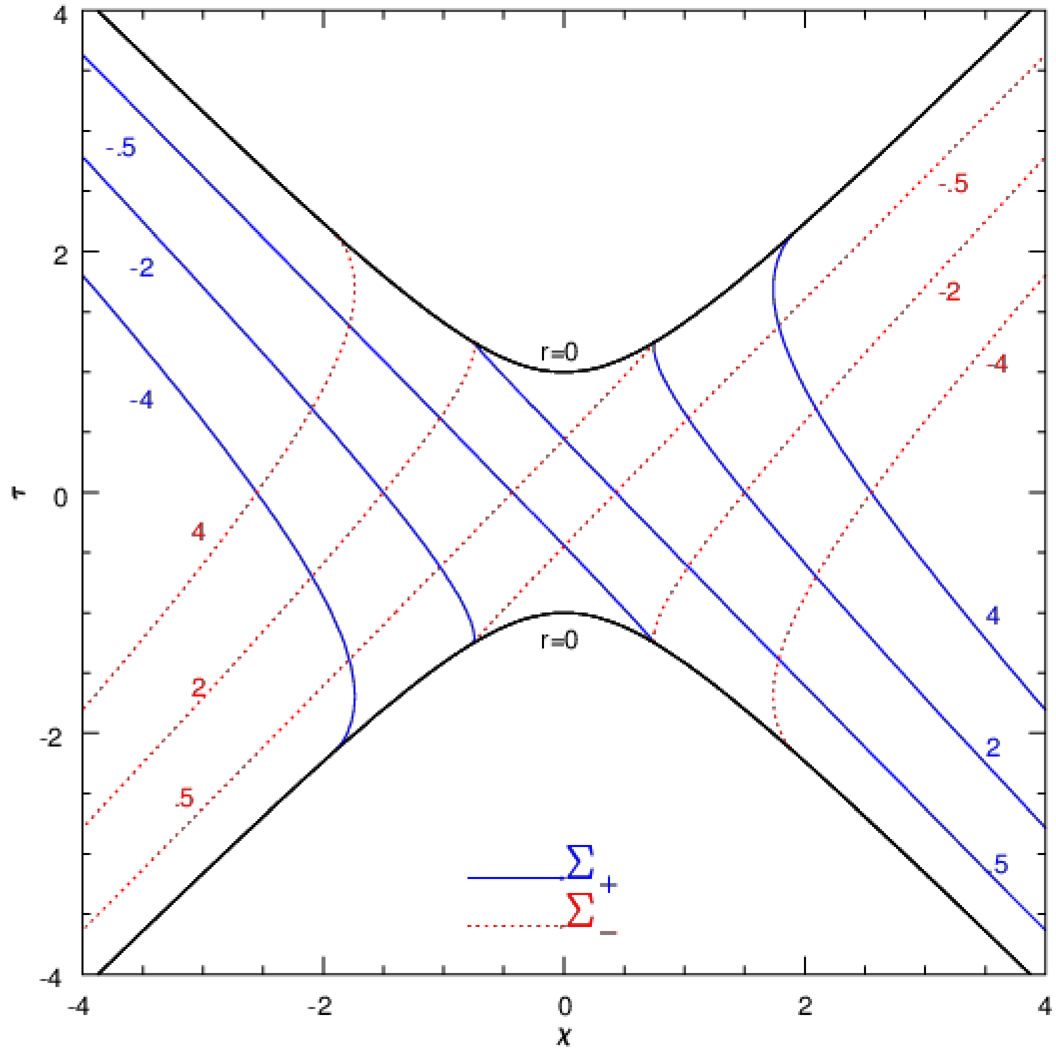


Each surface is a spatially flat surface

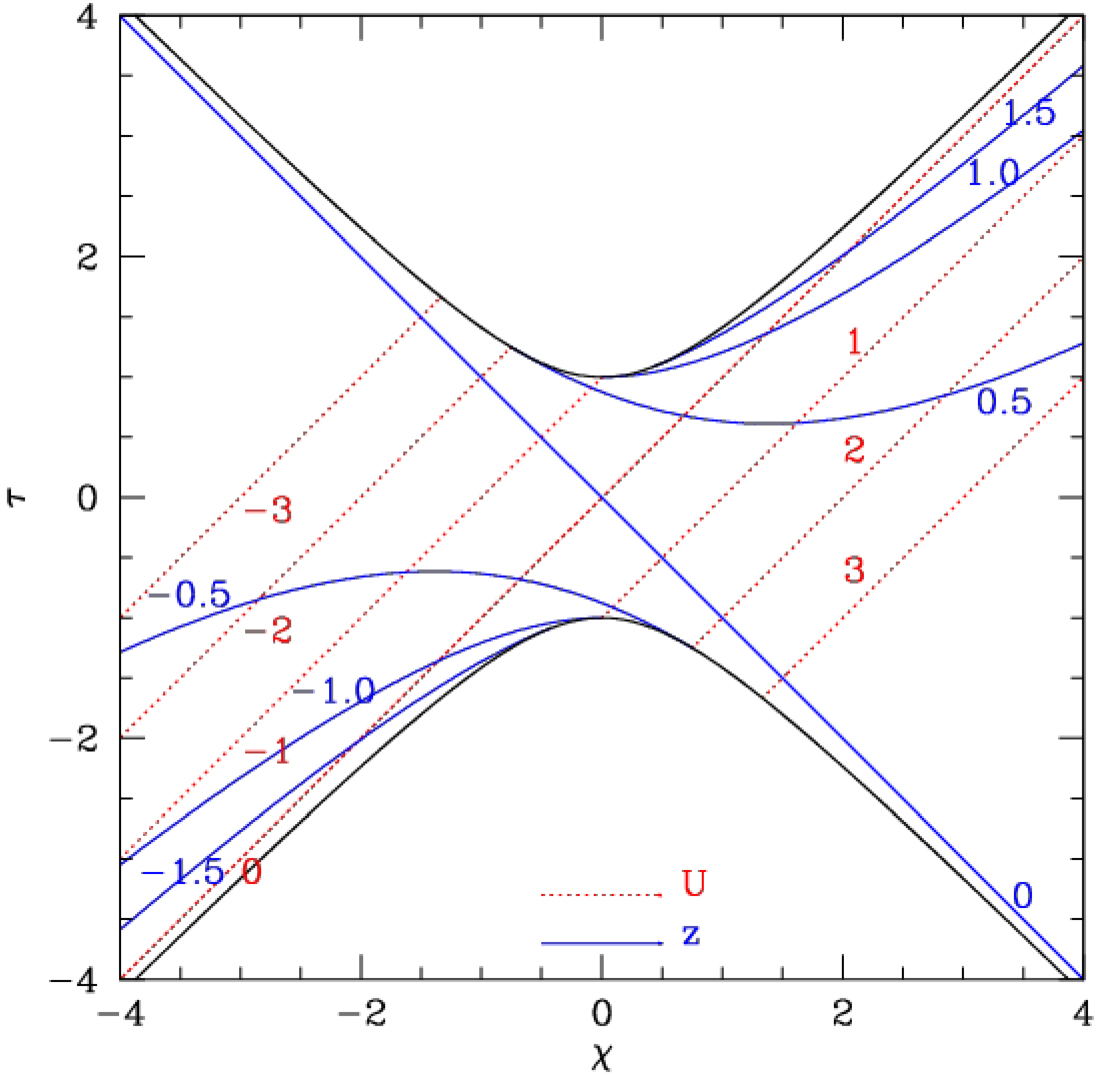
Synge coordinates



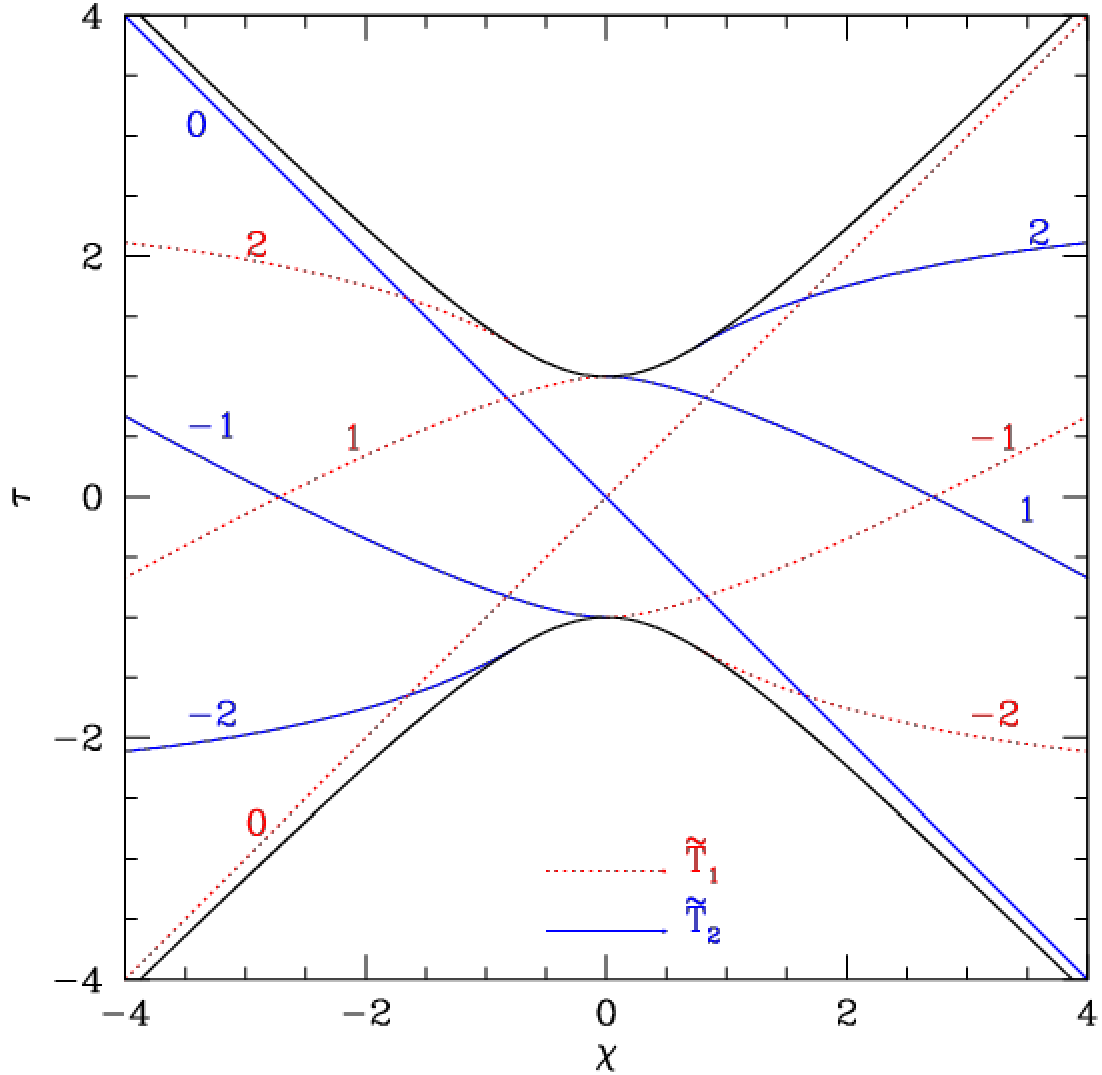
Lemaitre Coordinates.



Israel Newman coordinates



Eddington Finkelstein constant time surfaces



Trying to understand the Schwarzschild singularity (r=2M) was a major goal in the early days of General Relativity.

From our viewpoint now, that understanding was so so close at hand, and many people had all of the tools needed to understand that horizon. But somehow they all failed until 30-40 years after that solution was found.

One of the puzzles is why? When they were tripping over the answers, why did they fail to see them? Especially Einstein who continually said the right things, and then wandered off onto tangents which took him away.
At the very latest, Einstein and Rosen should have had the Synge solution 15 years before Synge.

What are we not seeing today?