Singularities and Divergences

Philosophical Lessons from Condensed Matter Physics

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Outline

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Let me begin with a quote from Michael Fisher's review article entitled *Scaling, Universality, and Renormalization Group Theory*. In a section entitled "The need for models" Fisher expresses what, from a contemporary philosophical perspective, is a rather heretical point of view.

"The traditional approach of theoreticians, going back to the foundation of quantum mechanics, is to run to Schrödinger's equation when confronted by a problem in atomic, molecular or solid state physics! One establishes the Hamiltonian, makes some (hopefully) sensible approximations and then proceeds to attempt to solve for the energy levels, eigenstates and so on. However, for truly complicated systems in what, these days, is much better called "condensed matter physics," this is a hopeless task; furthermore, in many ways it is not even a very sensible one!"

"The modern attitude is, rather, that the task of the theorist is to *understand* what is going on and to elucidate which are the crucial features of the problem. For instance, if it is asserted that the exponent α depends on the dimensionality, d, and on the symmetry number, n, but on no other factors, then the theorist's job is to explain *why* this is so and subject to what provisos. If one had a large enough computer to solve Schrödinger's equation and the answer came out that way, one would still have *no understanding* of why this was the case!" [5, pp. 46–47]

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- Fisher's primarily pragmatic reflections and insights have consequences well beyond solid state physics.
- Their development has consequences for understanding various relations between physical theories.
- They are crucial for developing a proper philosophical methodology—one that pays attention to the actual explanatory practice of physicists.

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- Need to get a handle on what Fisher means by the "crucial features" of a problem.
- Need to see what is involved in "elucidating" or understanding these crucial features.
- One conclusion: Appeal to mathematical singularities and the divergence of various quantities plays a fundamental role in addressing these two issues/questions.
- As singularities and divergences reflect places where laws apparently breakdown, this poses a serious challenge to philosophical views of explanation and understanding that hold laws to play the primary explanatory role.

- The "modern" attitude is motivated by the need to explain the universality that one sees in behaviours of systems. If only one (or just a few) systems had their order parameters at criticality scale as $|t|^{\beta}$, perhaps the best one could do is try to explain that similarity on the "traditional approach"—running to the Schrödinger equation and providing a from-first-principles account.

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- But the world displays a multitude of patterns—repeatable, reproducible phenomena—that appear despite wide variations in details.

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- One crucial feature: the explanation proceeds by a *mathematical* analysis of features of an abstract space—the space of hamiltonians or coupling constants.
- From a philosophical perspective, I think this is extremely important.

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- Classical examples include hydrodynamics/Navier-Stokes theory, and thermodynamics.

David Nelson— a "generalized notion of a fixed point."

"It turns out that not just critical points but entire phases of matter are described by a 'universal,' coarse-grained, long-wavelength theory. . . . One can make similar statements about the hydrodynamic laws derived for fluids in the nineteenth century. Upon systematically integrating out the high-frequency, short-wavelength modes associated with atoms and molecules, one should be able to arrive at, say, the Navier-Stokes equations. . . . Ignorance about microscopic details is typically packaged into a few phenomenological parameters characterizing the 'fixed point,' such as the density and viscosity of an incompressible fluid like water in the case of the Navier-Stokes equations." [7, p. 3.]

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- But this is not, by any means, how the equations of hydrodynamics were actually developed in the Victorian era.
- Instead, one begins with phenomenological patterns and tries to mathematically capture the regularities displayed by these patterns.

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- Obviously, the crucial features are whatever is essential to the existence of *robust* universal features—robust phenomenological patterns.
- In fact, I want to argue that in many cases what is crucial is not any special *physical* feature but rather, it is an essential bit of mathematics—an appeal to limits with sometimes striking singularities.

In condensed matter theory (broadly construed so as to include the physics of everyday stuff such as rainbows and dripping faucets) explanations of patterns or of "crucial features," or, to use G. I. Barenblatt's phrase ([1]), of the "principal laws or basic features" are provided by *asymptotic investigations*.

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- The modern theorist's explanation or *elucidation* of such crucial features employs asymptotic analysis.
- How does this actually work?

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- So, every "explanation" in the traditional from-first-principles derivation will be distinct—different molecular structure, etc.
- What can possibly unify these distinct from-first-principle stories? What do they all have in common?

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- This is what the RG does (explicitly) in the case of critical phenomena.
- It is the antithesis of the traditional philosophical explanatory strategy where providing causal, microscopic details is essential.

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- **Q:** How is it possible, according to physical theory, that such diverse systems can display the same behaviour at criticality?
- **A:** *A priori*, (it seems) the answer must be that the microscopic degrees of freedom that distinguish the different systems are (virtually) irrelevant for the behaviour of interest.
- This is possible when there is a loss of characteristic scale as when, at criticality, the correlation length among lattice spins diverges in the thermodynamic limit.

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- Most often this is accomplished through variable reduction—a key implement in the applied mathematician's tool box—that allows for the freezing out of various degrees of freedom.
- The RG is, in effect, one method among many that relies upon this mathematical tool.

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- In many/most instances one finds singularities in such reduced variable limits.
- Rather than something to be avoided, these singularities and divergences provide the loss of scale necessary to derive similarity or scaling laws that express the universal behaviour one is trying to understand.

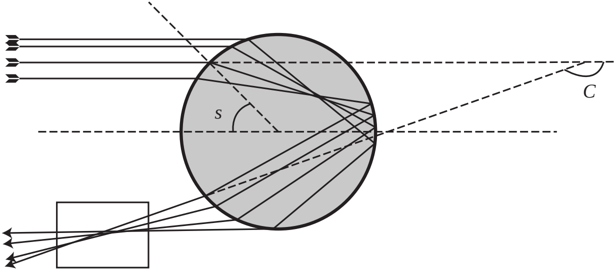

Figure: Rays Forming a Caustic Upon Refraction and Reflection in a Raindrop

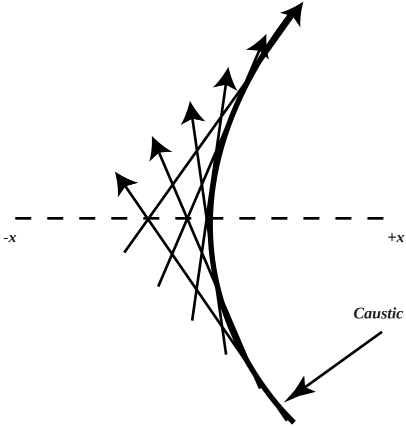

Figure: Fold Caustic

$$Ai(x) \equiv \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt. \quad (1)$$

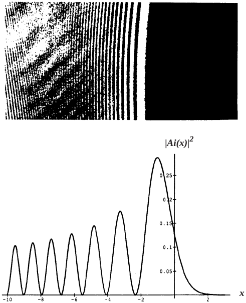

Figure: Airy Function and Supernumerary Bows

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- For the fold or rainbow the "diffraction catastrophe integral" is the following:

$$\Psi_{\text{fold}}(R) = \frac{k^{1/6}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{[i\frac{s^3}{3} + k^{\frac{2}{3}}Rs]} ds \qquad (2)$$

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  [compare (1)]
- So, for the fold, the exponents $\beta = 1/6$ and $\sigma_1 = 2/3$ and hence as $k \to \infty$ the intensity, $|\Psi|^2$, near the caustic rises to $\mathcal{O}(k^{1/3})$ and the fringe spacings on the bright side of the caustic are of $\mathcal{O}(k^{-2/3})$.

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- Of major importance is the fact that the fold caustic is structurally stable under diffeomorphism—it doesn't matter very much at all what the shapes of the raindrops are.
- The individual functions that characterize their shapes can all be shown to be in transformable into the normal form of the fold catastrophe:

$$\Phi_{\text{fold}}(s, \mathbf{R}) = \frac{s^3}{3} + \mathbf{R}s.$$

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- Here too, it is the *mathematical* divergence in intensity and other nonanalytic behaviour in the $k \to \infty$ limit that allows for a stability analysis (Thom's catastrophe theory) that explains the universality of the scaling law.

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    1. These explanations do not proceed from first principles derivations.
    2. These explanations involve *essential* appeals to singularities and divergences—places where laws break down.
    3. These explanations involve idealizations essentially: No divergence in correlation length without the thermodynamic limit.
    4. Idealizations are false, and orthodoxy has it that explanations must proceed from truths.

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- One will see no thermodynamic phase transitions without the thermodynamic limit.
- Most important is the fact that the singularities play ineliminable roles in these asymptotic understanding of these qualitative changes.

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- This has implications, in general, for intertheoretic relations.
- Foundational theories can be (and often are) explanatorily inadequate. They must appeal to idealized limits and singular (non-law-governed) behaviour. *This is strange.* It reminds me of a bit of dialog I once read.

The Boscombe Valley Mystery

Watson: "That sounds a little paradoxical."
Holmes: "But it is profoundly true. Singularity is almost invariably a clue." [4, p. 160]

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   - These phenomena do not admit of straightforward RG/asymptotic explanations.
   - They resist explanation by demonstrating the irrelevance of various details: Eg., they exhibit memory effects.
   - They fail to exhibit a clear separation of scales between the microlevel, the level of the single grain, and the level of bulk behaviour.

3. Granular media seem to require various, often incompatible, modeling and explanatory strategies. Can we find some way of unifying these different theoretical and explanatory approaches? What more can we learn about interlevel/intertheory relations?

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