“Irreversibility and the Landau-Boltzmann equation”

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1. The Boltzmann equation and condensed matter physics

2. Boltzmann's equation and H-theorem (1872)

2. The Landau-Boltzmann Equation

3. Some old and new problems

1. The Boltzmann Equation and condensed matter physics

- Boltzmann Equation describes evolution of (classical) dilute gases, in particular their irreversible approach to equilibrium.
- Condensed matter physics (e.g. normal Fermi liquids) is concerned with strongly interacting quantum particles.
- How could any version of the BE be applicable here?
- Rough answer: In a normal Fermi liquid at low temperatures, nearly all degrees of freedom are "frozen" in the Fermi sea. Only very few excitations ("quasiparticles") with energies near the Fermi surface contribute to the thermal properties of the system.
- Landau's theory: a (normal) Fermi liquid at low temperature can be seen as a dilute quasiparticle gas.

1. Boltzmann's equation and H-theorem (1872)

Assumptions:

1. The gas consists of a large number N of hard elastic spheres. It is so dilute that only binary collisions occur.
2. The state of the gas at time t may be represented by a distribution function f_t such that:

$$f_t(\vec{q}, \vec{p})\delta\vec{q}\delta\vec{p} \;=\; \tfrac{1}{N}\,(\text{number of particles with positions in } \delta\vec{q} \text{ and momenta in } \delta\vec{p}.)$$

3. the *Stosszahlansatz*: At every time t, the number of pairs of particles with variables $(\vec{q}_1, \vec{p}_1) =: 1$ and $(\vec{q}_2, \vec{p}_2) =: 2$ that are about to collide in the interval $[t, t + \delta t]$ is

$$\frac{N^2}{m} f_t(1) f_t(2) \|\vec{p}_1 - \vec{p}_2\| \delta t \delta\Omega$$

- From these assumptions, he derived the Boltzmann Equation,

$$\frac{\partial f_t(1)}{\partial t} + \frac{\vec{p}_1}{m} \cdot \frac{\partial f_t(1)}{\partial \vec{q}_1} - \vec{F} \cdot \frac{\partial f_t(1)}{\partial \vec{p}_1} = \frac{N}{m} \int \left(f(1')f(2') - f(1)f(2) \right) \|\vec{p}_2 - \vec{p}_1\| d\Omega d\vec{p}_2$$

  where $\vec{F}$ denotes an external force and (1'), (2') are the variables immediately after collision, and $\vec{q}_1 = \vec{q}_2 = \vec{q}_1' = \vec{q}_2'$
- Assuming this equation holds all t, one obtains the so-called H-theorem: the expression

$$H[f_t] := \int f_t(\vec{q}, \vec{p}) \ln f_t(\vec{q}, \vec{p}) d\vec{q} d\vec{p}$$

  is monotonically decreasing in time:

$$\frac{dH[f_t]}{dt} \leq 0$$

- Boltzmann identified $-kNH$ with entropy, and claimed he had thus given a strict dynamical proof of the Second Law.

- Boltzmann did not draw attention to the Stoszahlansatz:

  "The determination [of the number of collisions] can only be obtained in a truly tedious manner, [...] But since this determination has, apart from its tediousness, not the slightest difficulty, nor any special interest, and because the result is so simple that one might almost say it is self-evident I will only state the result. (1872, p. 323)

- Loschmidt's reversibility objection: Take any motion in which H decreases, and reverse the velocities of all particles in the final state. We will obtain a motion for which H is increasing.
- By focusing on the number of particles that will collide in δt, —and not demanding the same thing about particles that just have collided,— a time-asymmetrical, and non-dynamical ingredient is introduced.

- Boltzmann recognized the importance of the Stosszahlansatz in the 1890s, and tried to motivate it by an asumption of "Molecular Chaos".
- Modern views in kinetic theory (following Jeans 1902) often take an ensemble over phase space and define marginal distributions:

$$f_t^{(1)}(\vec{q}_1, \vec{p}_1) := \int \rho_t(\vec{q}_1, \vec{p}_1; \ldots ; \vec{q}_N, \vec{p}_N) d\vec{q}_2 d\vec{p}_2 \cdots d\vec{q}_N d\vec{p}_N$$

$$f_t^{(2)}(\vec{q}_1, \vec{p}_1; \vec{q}_2, \vec{p}_2) := \int \rho_t(\vec{q}_1, \vec{p}_1; \ldots ; \vec{q}_N, \vec{p}_N) d\vec{q}_3 d\vec{p}_3 \cdots d\vec{q}_N d\vec{p}_N$$

They define 'Molecular Chaos' as

$$f_t^{(2)}(\vec{q}_1, \vec{p}_1; \vec{q}_1, \vec{p}_1) = f_t^{(1)}(\vec{q}_1, \vec{p}_1) f_t^{(1)}(\vec{q}_2, \vec{p}_2)$$

Unfortunately, this assumption is by itself is invariant under time-reversal.(cf. Huang (1963), Lanford (1975), Cercigianni, Illner & Pulverenti (1994))

2. The Landau-Boltzmann Equation for normal Fermi liquids

- Aim: find a quasi-particle density: $n_t(\vec{q}, \vec{p})$ (ignoring spin variable). Compatible with Uncertainty Principle?
- More systematically, consider system of N fermions in non-relativistic second quantisation formalism, in Heisenberg picture. Define a Green function

$$G(1,2) = \mathrm{Tr}\rho\, \psi(\vec{r}_1, t_1)^\dagger \psi(\vec{r}_2, t_2) = \langle \psi(\vec{r}_1, t_1)^\dagger \psi(\vec{r}_2, t_2) \rangle$$

- Perform Wigner transform to obtain:

$$n_t(\vec{q}, \vec{p}) = \frac{1}{2\pi} \int d\vec{r} e^{-i\vec{p}\cdot\vec{r}} \langle \psi^\dagger (q - \frac{r}{2}), t) \psi(q + \frac{r}{2}, t) \rangle$$

as the analogon of Boltzmann's $f_t(\vec{q}, \vec{p})$. However, Wigner transform is not necessarily non-negative...

- Assume that quasiparticle potential is short-range and apply Born collision approximation. The result is a Boltzmann-like equation with left-hand-side (streaming terms)

$$\frac{\partial n_t}{\partial t} + \nabla_{\vec{p}}\epsilon \cdot \frac{\partial n_t}{\partial \vec{q}} - \nabla_{\vec{q}}\epsilon \cdot \frac{\partial n_t}{\partial \vec{p}} = * \qquad (1)$$

and right-hand side (collision integral):

$$* = \int d\vec{p}_2 d\vec{p}_1' d\vec{p}_2' W(1', 2'; 1, 2) \left[n(1')n(2')(1 - n(1))(1 - n(2)) - n(1)n(2)(1 - n(1')(1 - n(2') \right]$$

(where W is the scattering probability and all n's evaluated at the same value for position: $\vec{q}_1 = \vec{q}_1' = \vec{q}_2 = \vec{q}_2'$)

- If we define

$$H[n_t] := \int (n_t \ln n_t + (1 - n_t) \ln(1 - n_t)) \, d\vec{q} d\vec{p}$$

the LB equation implies (cf. Belic 1997):

$$dH/dt \leq 0$$

3. Some old and new problems

New features:

- The Landau-Boltzmann equation contains spin as extra variable.
- The velocity is replaced by $\nabla\epsilon$ where the quasiparticle energy ϵ itself depends on $n_t(\vec{q}, \vec{p})$
- Instead of Boltzmann's Stosszahlansatz, the probability for a collision $(1, 2) \longrightarrow (1', 2')$ is now proportional to four terms, i.e. the probabilties of state (1) ,(2) being occupied, while (1') and (2') are non-occupied. This is due to the Pauli exclusion principle.

Old problems

- Reversibilty objection: QM is just a time-reversal invariant a classical Hamiltonian mechanics. Hence the LB equation cannot hold universally.
- An alternative definition of entropy $S = -Tr\rho \ln \rho$ remains constant in time for any ρ.

New problems:

- -Wigner transform is not guaranteed to be non-negative. But if n is not always and everywhere positive, then definition of entropy becomes complex-valued!
  This problem may perhaps be alleviated by choosing a different transform (e.g. Husimi distribution?).
- Belic explains the "crucial assumption", i.e. the analogy to the Stosszahlansatz for collision $(1, 2) \longrightarrow (1', 2')$ sas follows.
  "The actual factor should be the simultaneous probability of the states 1,2 being occupied and 1',2' being empty. To treat these as independent is an approximation valid to leading order at low quasi-particle density."
  But this assumption is equivalent to the claim that the probability for 1,2 being empty and 1',2' occupied are also independent!
  This means one cannot argue for this assumption by saying that states before a collision are independent but not after.

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