Random walks, Brownian motion, and percolation

Martin Barlow

[1]Department of Mathematics, University of British Columbia

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Two models in probability theory

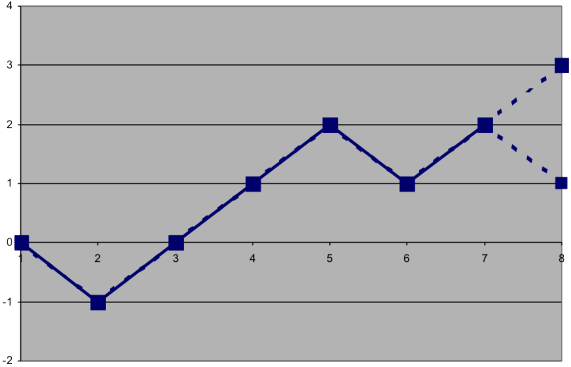
In this talk I will discuss two areas of probability theory:

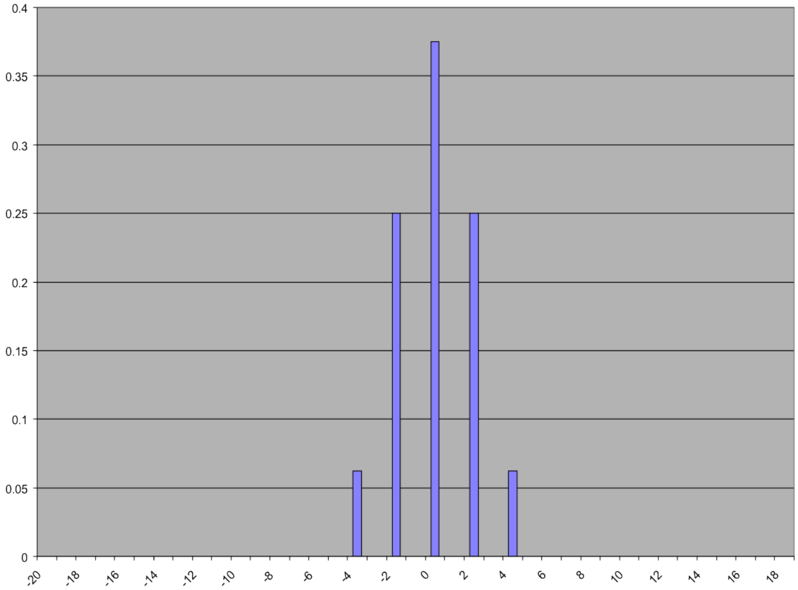
Random walks. Introduced early 1900s.
The first generation of problems have mainly been solved.
For the second generation of problems, from the 1980s, some are still open (i.e. unsolved).

Percolation. Introduced in late 1950s.
In spite of 60+ years of work, some basic problems are still unsolved.

Random walks

The basic idea is of a particle moving 'at random'/ in space. It starts at 0, and each time step (second) we toss a fair coin. If the coin is heads then the particle moves up by 1 unit, if the coin is tails then it moves down one.

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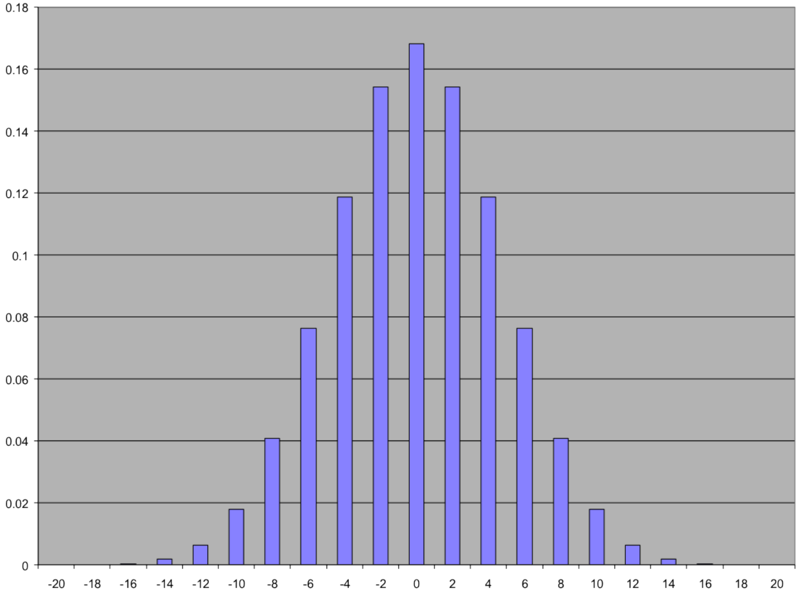
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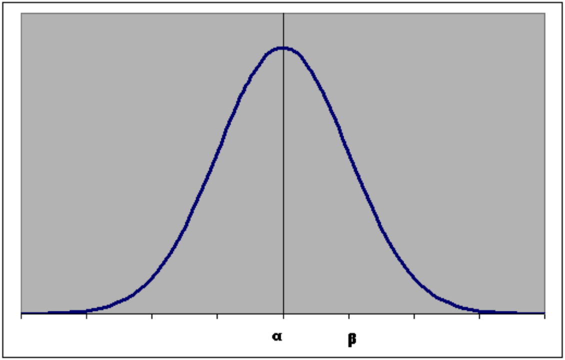
Normal distribution

After a large number of steps the probabilities are close to the Normal distribution ('bell curve'):

$$f(x) = \frac{1}{\sqrt{2\pi\beta^2}} e^{-\frac{(x-\alpha)^2}{2\beta^2}}.$$



RW after a large number of steps

Since heads and tails are equally likely the average value (mean) is zero.

Central Limit Theorem (de Moivre 1738, Laplace, 1812)
Let X_n be the position of the random particle after n steps. Then

$$\mathbb{P}\left(\frac{X_n}{\sqrt{n}} \leq t\right) \to \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds.$$

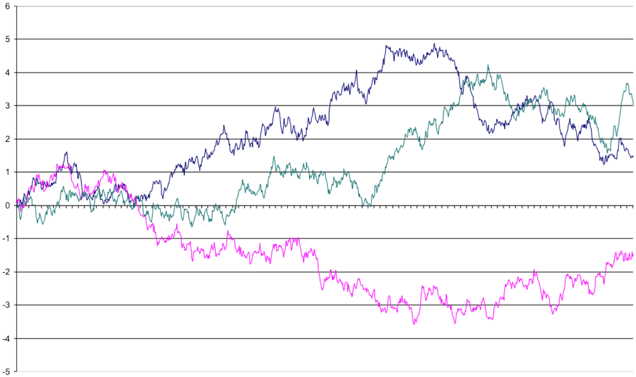
It follows than after n steps the random walk is very unlikely to be be less than $-3\sqrt{n}$ or greater than $3\sqrt{n}$.

The RW is unlikely to be exactly zero: for large n

$$\mathbb{P}(X_{2n} = 0) \sim \frac{1}{\sqrt{\pi n}}.$$

Brownian motion

If we rescale the RW so that it takes very small jumps very frequently, then we get 'Brownian motion':



Brownian motion in 2 or more dimensions

The previous slides showed a random walk in one dimension ($d = 1$): the particle could only move up or down.
But one can also have a RW which moves in two dimensions: i.e. each step in goes N,S,E or W each with probability $\frac{1}{4}$
Three dimensional random walk moves in one of 6 directions (up, down, N,S,E,W) each time step. Rescaling gives Brownian motion in 2, 3 dimensions.

Random walk in d dimensions:

$$\mathbf{X}_n = (X_{1,n}, X_{2,n}, \ldots, X_{d,n}).$$

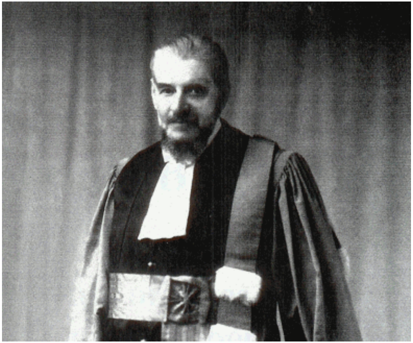
At each time step, choose an index i at random from $\{1, \ldots, d\}$ and then move $\mathbf{X}$ by taking

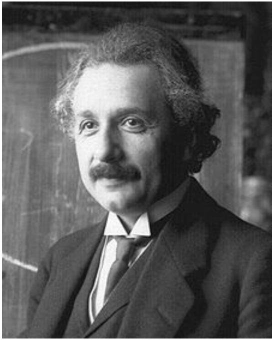
$$X_{i,n+1} = X_{i,n} \pm 1.$$

Early work on Brownian motion

Louis Bachelier (1870-1946): 1900 thesis ‘Theorie de la Speculation’. Used Brownian motion to model stock and option prices.

Albert Einstein (1879-1955): physics papers in 1905, 1906.





Brown and Brownian motion

The botanist Robert Brown (1827) observed through the microscope random motion in minute particles suspended in water. Einstein showed this was due to random collisions by the surrounding (much smaller) water molecules.

He was able to use this to calculate how rapidly a dissolved substance (e.g. sugar) will spread out in water.

He also showed that the probabilities for the random walk/Brownian motion obey the mathematical equation for heat flow, the 'heat equation':

$$\frac{\partial u}{\partial t} = \nabla^2 u.$$

Bachelier's work

Bachelier used random walks and Brownian motion to model stock prices. He then used this to calculate the value of options.

Though his work was not neglected, it did not receive the recognition it deserved.

Brownian motion as a model for stock prices was used by Fisher Black (1938–1995) and Myron Scholes (1941–) to obtain their option pricing formula (1973).

(Scholes received the Nobel Memorial prize in Economics for this work in 1997.)

Open problems for random walks

Suppose two random walks are started a distance 1 apart, and allowed to run for a large number n steps. What is the probability that the paths they trace out don't cross? If this is roughly

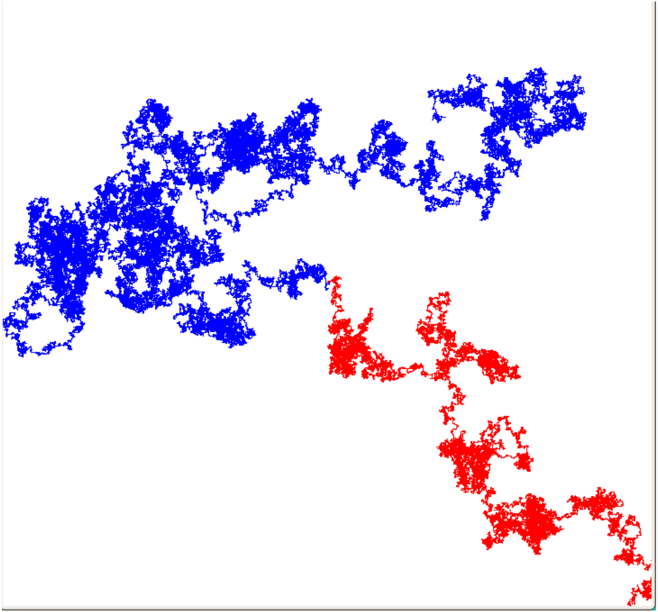
$$\frac{1}{n^{\zeta}}$$

then we say the 'self intersection exponent is' ζ. In 2 dimensions this has been proved to be 5/8. (Lawler, Schramm, Werner).

Not known in 3 dimensions – simulations give about 0.29.

If $d \geq 5$ then paths of two independent random walks will with positive probability fail to intersect, so $\zeta = 0$.

Two non-intersecting random walks



Fractals and dimension

Brownian motion is 'statistically self similar' – if one looks at a small part of the picture magnified, it will look similar to the larger picture.

The French mathematician/physicist B. Mandelbrot introduced the word 'fractal' to describe objects of this kind, and gave examples (coastlines, clouds, trees) of fractal-like objects arising in nature.

Fractal dimension. This extends our usual idea of dimension to fractal type objects. If one needs $N(r)$ small cubes/squares of side r to cover an object, then it has dimension d_f if

$$N(r) \approx \frac{1}{r^{d_f}} \quad \text{as } r \to 0.$$

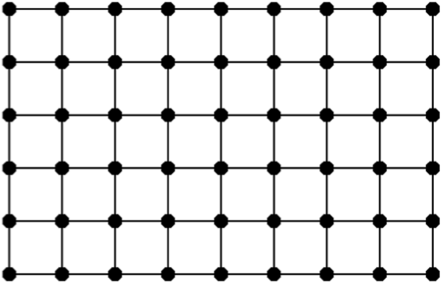
Mandelbrot conjectured that the dimension of the 'Brownian coastline' was $4/3$ – proved by Lawler, Schramm and Werner.

Percolation

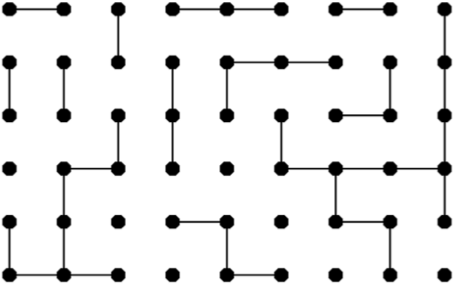
This model was introduced by Broadbent and Hammersley in 1957.
Broadbent was working on gas masks for use in coal mines.
Hammersley said:

“These masks contained porous carbon into which the gas could penetrate. The pores constituted a random network of tiny interconnecting tunnels, along which the gas could move. If the pores were richly enough connected, the gas could permeate the carbon, but if not then the gas would not get beyond the surface. So there was a critical point, above which the mask worked well, and below which it was ineffective.”

One starts with a complete network consisting of vertices (nodes) and bonds.



Choose a probability p and keep each bond with probability p. (Here $p = 0.4$).

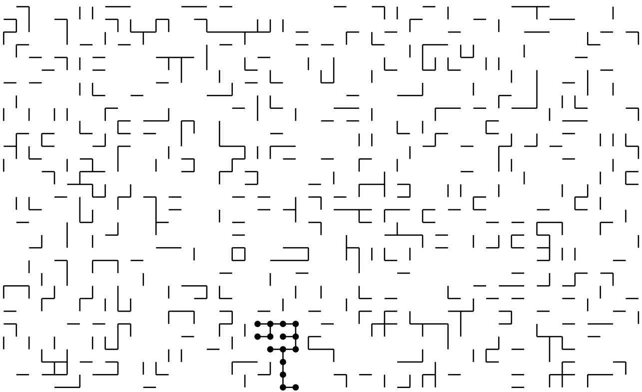


Universality

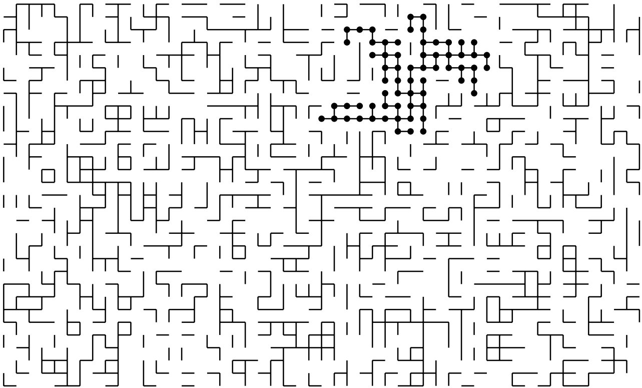
This is not a very realistic model of the gas mask. In fact it is what is called a ‘toy model’ – a greatly simplified model which aims to capture some key features of the real life situation.

It has been found that even toy models of this kind (in statistical physics) are very hard to solve. In fact they include unsolved problems as hard and as important as any in mathematics.

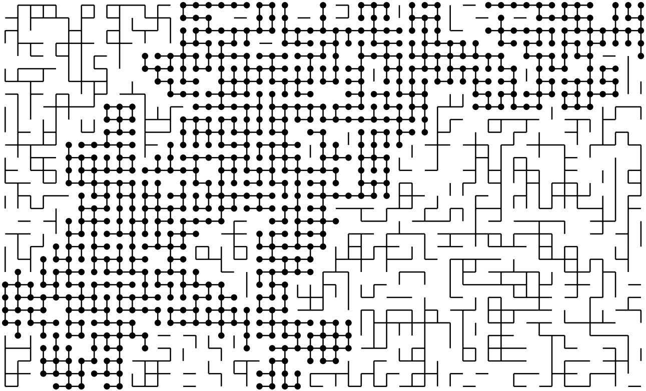
Physicists believe many of these toy models have the property of ‘universality’ – if one can work out the behaviour of the toy model then the real life situation behaves in the same way.



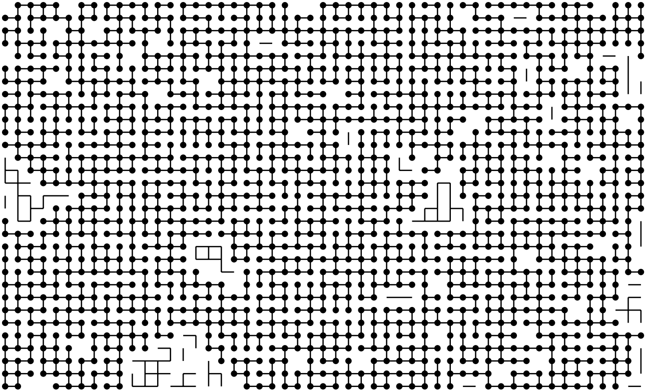
$p = 0.2$, largest cluster marked



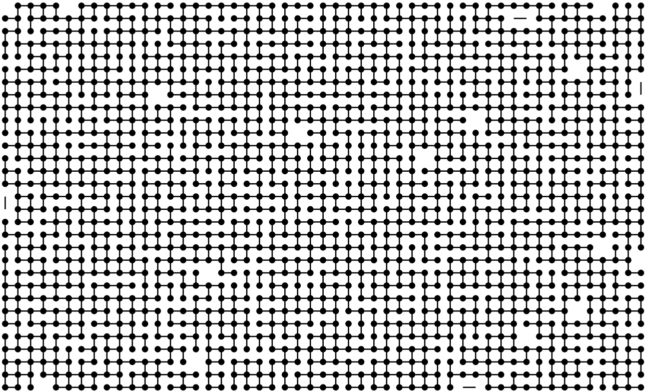
$p = 0.4$



$p = 0.5$



$p = 0.6$



$p = 0.8$

Phase transition

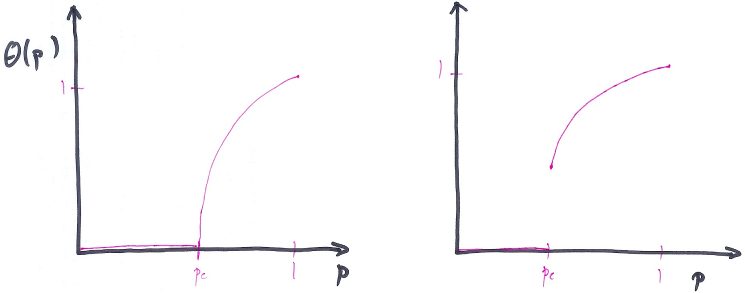
One can consider percolation on other lattices, and in any number of dimensions.

The collections of nodes connected by bonds are called 'clusters'. Percolation has a 'phase transition':
(1) If p, the probability a bond is retained, is small, then all the clusters are finite, and nearly all are small. (The 'subcritical regime').
(2) If p is large then there is (with probability one) a giant cluster denoted $\mathcal{C}_\infty$ which extends infinitely far in all directions. (The 'supercritical regime').

This regime change occurs sharply at a critical probability denoted p_c.
Set

$$\theta(p) = \mathbb{P}_p(0 \text{ is in the infinite cluster } \mathcal{C}_\infty).$$

Main open problem for percolation



How does $\theta(p)$ behave at $p = p_c$?

Main open problem for percolation

Are there infinite clusters at the critical point, i.e. if $p = p_c$? It is believed that the answer is 'No' in all dimensions.

Proved for $d = 2$ by Kesten (1980).

Proved for $d \geq 19$ by Hara and Slade (1990). (Extended recently to $d \geq 12$ or so.)

The Hara-Slade approach cannot work if $d \leq 6$, and no known methods can touch this problem in 3 dimensions.

Scaling exponents. It is further conjectured that for $p > p_c$

$$\theta(p) \approx (p - p_c)^{\beta}, \text{ as } p \to p_c,$$

and the exponent β is believed to be 'universal', i.e. depending only on the dimension d.

Disease contact networks

Percolation is also used in models of the spread of infectious diseases.

The nodes are people, and one has bonds between two people if they have contact. ('Contact networks'). If the disease has probability p of being transmitted then the clusters in the new network represent the people who will be infected if one person gets the disease.

If the percolation is subcritical one infected person only infects a small number of people. If the percolation is supercritical then one person will infect a large number of people.

Contact networks are very different from the Euclidean lattice shown above – they have the 'small world' property.

Random walks and percolation

Percolation provides models for disordered materials. Since Einstein we have known that there is a connection between random walks and heat conduction. So, random walks on percolation clusters should tell us about heat conduction in disordered media.

This problem was suggested by the French physicist Pierre-Gilles de Gennes (1932-2007) in a 1976 article in ‘La Recherche’. He called it the problem of the ‘ant in the labyrinth’.

La percolation : un concept unificateur

par Pierre Gilles de Gennes

Cliché Henri Cartier-Bresson / Magnum.

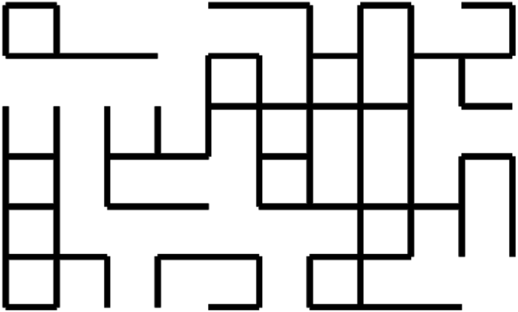
Pierre Gilles de Gennes, professeur au Collège de France et directeur de l'École de physique et chimie de Paris, est connu notamment par ses travaux sur

Proposé en 1956 par le mathématicien anglais Hammersley, le concept de percolation permet une description statistique des systèmes constitués d'un grand nombre d'objets qui peuvent être reliés entre eux. Dans un tel système, la communication à grande distance est soit possible soit impossible suivant le nombre d'objets et de liaisons : il existe un seuil de transition précis entre ces deux régimes.

Le problème des cheveux emmêlés. Nous savons tous qu'une chevelure exposée au vent devient difficile à peigner. Il est amusant de voir que cet enchevêtrement est associé à une transition de percolation. Ici, on dira que deux cheveux dont les racines sont voisines sont associés dans un même « amas » si l'un est enroulé au moins une fois autour de l'autre, ou s'ils forment des nœuds plus complexes. En présence d'un

The ant in the labyrinth

The random walked is the ‘ant’, and each time step it chooses a direction at random from those available to it.



First problem – slow down

Does removing bonds 'slow down' the random walk? In the Euclidean lattice (i.e. $p = 1$) one has, writing X_n for the position of the r.w. at time n,

$$E(X_n - X_0)^2 = \langle (X_n - X_0)^2 \rangle = n.$$

For the r.w. on the percolation cluster, X'_n does one have

$$E(X'_n - X'_0)^2 = \langle (X'_n - X'_0)^2 \rangle \leq n?$$

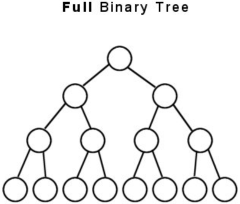
One can also ask about non-random removal of bonds – i.e.

Question. *Starting with the Euclidean lattice, if one removes some bonds, does one have*

$$E(X'_n - X'_0)^2 \leq n?$$

Answer. NO. One can remove bonds so as to 'speed up' the random walk. (But not by very much.)

RW on lattice with bonds removed.



If a random walk is started at the top of the tree, it moves downwards with roughly linear speed, since at each time step it has probability $\frac{2}{3}$ of going down, and $\frac{1}{3}$ of going up.

There is not 'enough room' in the two dimensional lattice to put in a full binary tree $\mathbb{B}$. But one can put in small bits of $\mathbb{B}$ so that one has, for certain n,

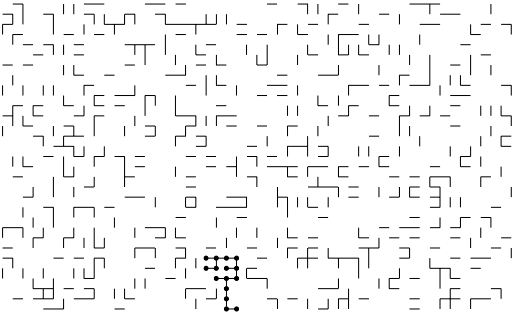
$$E(X_n' - X_0')^2 \geq n(\ln n).$$

construction, so is unlikely to occur for percolation.

RW on percolation clusters

There are three regimes.

Subcritical, i.e. $p < p_c$. There are only finite clusters, so there is not much to say about the random walk.



$p = 0.2$, largest cluster marked

Supercritical, i.e. $p > p_c$

In this case it is known that there is an unique infinite cluster $\mathcal{C}_\infty$.

If X is started in $\mathcal{C}_\infty$ then X converges to Brownian motion with diffusivity $D(p)$, where

$$0 < D(p) \le D(1).$$

The SRW X also behaves like the SRW on $\mathbb{Z}^d$ in many other ways also. In particular, the return probabilities satisfy

$$p_{2n} = \mathbb{P}(X_{2n} = X_0) \sim \frac{c}{n^{d/2}} \quad \text{as } n \to \infty.$$

So what is called the *spectral dimension* of $\mathcal{C}_\infty$, given by

$$d_s(\mathcal{C}_\infty) = \lim_{n\to\infty} \frac{2 \ln p_{2n}}{\ln(1/n)}, \text{ is equal to } d.$$

Only one supercritical regime for heat

If p is close to 1, then $\mathcal{C}_\infty$ looks like $\mathbb{Z}^d$ with small defects.
If p is close to p_c then (e.g. in 3 dimensions) $\mathcal{C}_\infty$ looks like a 3 dimensional net: lots of irregular strands and 'dangling ends', but also lots of connections.

Although the geometry of the cluster seems very different in the two cases, there is no (known) qualitative difference between them from the point of view of heat conduction or the random walk.

It is possible that for more delicate equations, such as the Schrodinger equation, there is an additional phase transition, i.e. there exists

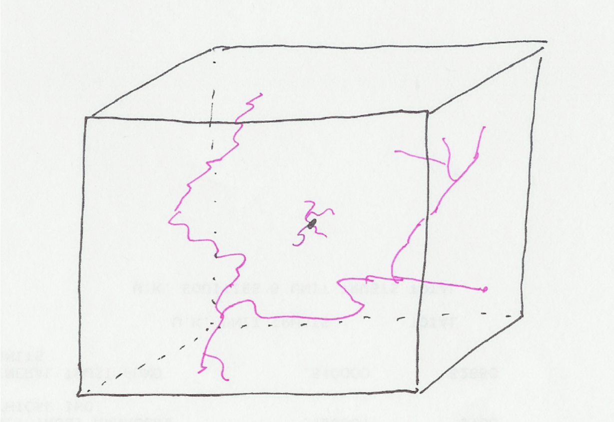
$$p_c < p_q < 1$$

with different behaviours for $p_c < p < p_q$ and $p_q < p < 1$.

Critical percolation

Asking about this means asking either about percolation with $p = p_c$, or as $p \to p_c$.

Conjectured situation at p_c. All clusters are finite, but any large box side n will contain finite clusters with diameter of order n.



Incipient infinite cluster

This is an infinite cluster $\tilde{C}$ which looks locally like the large finite clusters which (are believed to) occur at p_c.
Constructed when $d = 2$ (Kesten) and for large d (van Hofstad, Jarai).

Alexander–Orbach Conjecture (1983). For all $d \geq 2$,

$$d_s(\tilde{C}_d) = \lim_{n\to\infty} \frac{\ln p_{2n}}{\ln(1/n)} = \frac{4}{3}.$$

(Here as before p_{2n} is the return probability of the RW after $2n$ steps.)
This was a bold conjecture, but was at least partially supported by numerical evidence in 1983.

AO conjecture

Why did they think this could be independent of dimension?

A general idea in statistical physics is of 'upper critical dimension' d_c: for $d > d_c$ global phenomena cease to be dimension dependent.

For percolation $d_c = 6$. At $p_c(d)$ there is just enough probability of an edge being open to allow large scale connected structures to exist. However, these structures are 'thin' and when $d > d_c$ they don't self-intersect except locally – hence they don't 'see' the true dimension of the space they are in. In fact, they are close to being 'fractal trees', and so percolation in high dimensions should be similar to percolation on the binary tree.

It was known that for the IIC on the binary tree one had $d_s = 4/3$. The bold part of the AO conjecture was to guess this also held for $2 \le d \le 5$.

Kozma and Nachmias, 2009 (improving earlier work by MB, Kumagai, Jarai, Slade on a more complicated percolation model): the AO conjecure *is* true if $d \geq 19$.

It is not now expected to be true for $d = 2, 3, 4, 5$.

General situation for models in statistical physics.

(1) Trees are easy, and solutions can often be written down explicitly,

(2) Next easiest is Euclidean space in very high dimensions, which approximates the tree case,

(3) Next easiest is two dimensions, due to special properties and link with complex analysis,

(4) Hardest is the space we live in, 3 (and 4) dimensions.

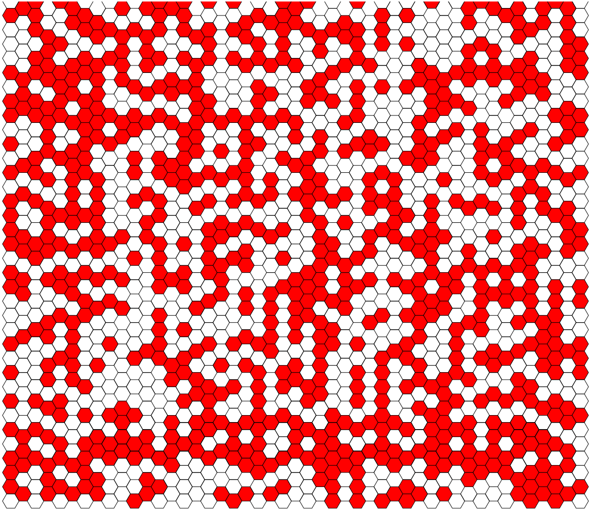
Two dimensions

Around 2000, Lawler, Schramm and Werner introduced a family of stochastic processes, now called Schramm-Loewner evolution or SLE_κ, which describe random interfaces in two dimensions.

The parameter κ lies between 0 and 16.

For example, the 'Brownian coastline' is an SLE process with $\kappa = 8/3$.

The interfaces of percolation clusters have been proved (by Smirnov) to be SLE curves (with $\kappa = 6$) for one particular percolation model – site percolation on the triangular lattice.



Using the SLE limit, many exponents have been calculated for the IIC for this percolation process. For example, the dimension of the cluster is 91/48. and the dimension of the boundary of the 'holes' is 7/4.

I do not know of any generally accepted conjecture on what the 'spectral dimension' d_s should be.

To calculate d_s one needs 'electrical resistance' properties of the IIC, and these are harder to obtain than the geometric properties which have been obtained from the SLE theory.

It is also likely that one needs to know the length of the shortest path in the cluster across a box of side n – another quantity that the SLE theory does not seem to give.