# Les Houches Lectures (1)

1) - SWT & its failure
- NL$\sigma$M:
    - top term
    - connection to spin chain
    - $\theta = 0$ case - $\beta$-fns, large-N

2) Integer-S Chains
- gap, susceptibility
- neutron scattering - near $q = \pi$, 0
- anisotropy

3) Half-Integer Case
- JW trans
- bosonization
- neutron scattering X-sec
- staggered field (?)

## Spin-Wave Theory & its failure in D=1

$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + \cdots \quad \text{(general D for now)}$$

- classical gnd. state $\begin{matrix} \uparrow & \downarrow \\ \downarrow & \uparrow \end{matrix}$ (bipartite lattice)
  $J > 0$, AF

$$H = J \sum \left[ S^z_i S^z_j + \tfrac{1}{2}\left( S^+_i S^-_j + S^-_i S^+_j \right) \right]$$

__not__ quantum gnd. state

$\cancel{[S^+, S]}$ $[S^a, S^b] = i\epsilon^{abc} S^c$

$\| [S^a, S^b] \| = S$ — $S \to \infty$ — classical limit

SWT — expand around ordered state

$S^z = S - a^+ a$ $\qquad$ $S^z = -S + b^+ b$

$S^- = \sqrt{2S}\, a^+ \left(1 - \frac{a^+ a}{2S}\right)^{1/2}$ $\qquad$ $S^+ = \sqrt{2S}\, b^+ \left(1 - \frac{b^+ b}{2S}\right)^{1/2}$

A - sub-lattice $\qquad$ B - sub-lattice

$H$ quadratic to $O(S)$ - Bogliubov transf.

- 2 identical branches $W_k = V_s |\vec{k}|$, $\vec{k} \to 0$ (or $\vec{k}_0$) Goldstone mode

$$\langle S^z_i \rangle_A = S - \int \frac{d^D k}{W_k} + O\left(\frac{1}{S}\right)$$

- finite correction - small at $S \gg 1$ except in $D=1$
  [ 40% for $S = \frac{1}{2}$ in $D=2$ ]

$$\langle \vec{S}_i \cdot \vec{S}_j \rangle = (-1)^{i-j} \left[ S^2 - \frac{2S}{\pi} \left( \ln|i-j| + \text{cnst} \right) + \cdots \right]$$

suggests $\xi \sim e^{c \cdot S}$

$e$ $\Delta \sim \frac{V}{\xi} \sim e^{-cS}$ ?

- only true for $S$ integer !

- need a large-S theory that doesn't assume ordered state in D=1 – NLσM.

$$\mathcal{L} = \frac{1}{2g}(\partial_\mu \vec{\varphi})^2, \quad \vec{\varphi}^2 = 1 \qquad (v=1)\ (+\text{top term})$$

- could also expand around ordered state

$$\vec{\varphi} = (\sqrt{g}\varphi_1, \sqrt{g}\varphi_2, \sqrt{1-g\varphi_1^2-g\varphi_2^2})$$

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{2}(\partial_\mu \varphi_a)^2 + \frac{g(\varphi_a\partial_\mu\varphi_a)(\varphi_b\partial_\mu\varphi_b)}{1-g\,\varphi_c\varphi_c}$$

$$\langle \varphi^3 \rangle = 1 - g\int \frac{d^2K}{K^2}$$ – same problem

$\Rightarrow$ no LRO in D=1

Hamiltonian density

$$\mathcal{H} = \frac{g}{2}\vec{l}^{\,2} + \frac{1}{2g}\left(\frac{d\vec{\varphi}}{dx}\right)^2$$

$$\vec{l} = \frac{1}{g}\vec{\varphi}\times\partial_0\vec{\varphi}, \quad \vec{l}\cdot\vec{\varphi} = 0$$

$$[l^a(x), l^b(y)] = i\,\epsilon^{abc}l^c(x)\,\delta(x-y)$$

$\Rightarrow \int dx\, l^a \equiv L^a$ normalized total spin operators

$$[l^a(x), \varphi^b(y)] = i\,\epsilon^{abc}\varphi^c(x)\,\delta(x-y)$$

$$[\varphi^a, \varphi^b] = 0$$

- relation to spin chain:

$$\vec{S}_j = (-1)^j S\,\vec{\varphi}(aj) + \vec{l}(aj)$$

- a long-wave length theory only (ie low energy $k = 0$ or $\pi/a$)

$\vec{\varphi}$ $\vec{l}$ vary slowly on lattice scale

$\vec{\varphi}$ = AF order parameter, $\vec{l}$ = conserved total spin density

- sub. in H, expand in derivatives to 2nd order – regarding $\vec{l}$ as 1st order gives $g = \frac{2}{S}$, $V = 2JS$, correct constraints & commutators

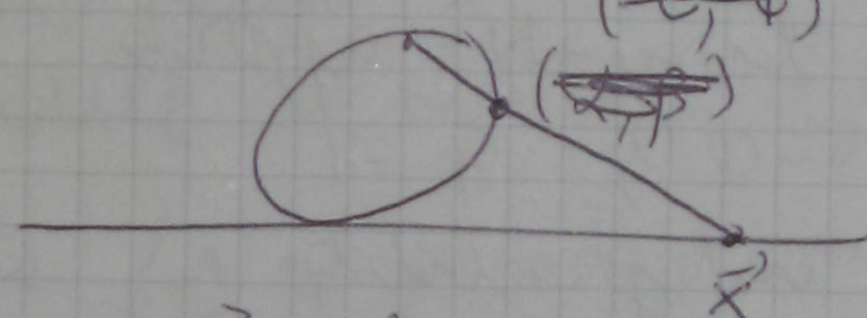
- we cannot do pert. th. in NLσM either but it is a convenient model in which to study non-perturbative effects
- top. term

$$\delta \mathcal{L} = \frac{\theta}{8\pi}\epsilon^{\mu\nu}\vec{\varphi}\cdot(\partial_\mu\vec{\varphi}\times\partial_\nu\vec{\varphi})$$ - total derivative $\Rightarrow$ no effect in pert theory

$$H \to \frac{g}{2}\left(\vec{l} + \frac{\theta}{4\pi}\frac{d\vec{\varphi}}{dx}\right)^2 + \frac{1}{2g}\left(\frac{d\vec{\varphi}}{dx}\right)^2$$

- this term also arises from spin chain with $\theta = 2\pi S$
- consider imaginary time Feynman path integral - b.c. $\varphi(\vec{x}) \to \vec{\varphi}_0$ at $|\vec{x}| \to \infty$
- can then project 2D space-time onto sphere

~~$(\theta, \varphi)$~~ $(\lambda, \eta)$



$\vec{\varphi}$ also is a pt on sphere

$$\vec{\varphi} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\beta)$$

$$\delta S = \frac{i\theta}{4\pi}\int d\lambda\, d\eta\, \sin\alpha\left(\frac{\partial\alpha}{\partial\lambda}\frac{\partial\beta}{\partial\eta} - \frac{\partial\alpha}{\partial\eta}\frac{\partial\beta}{\partial\lambda}\right)$$

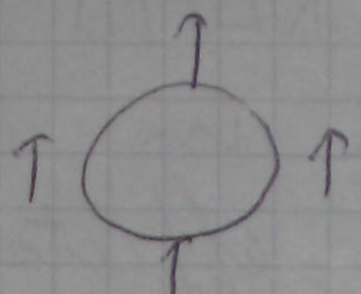
$$= \frac{i\theta}{4\pi}\int d\beta\, d\alpha\, \sin\alpha$$

$$= \frac{i\theta}{4\pi}\Omega$$

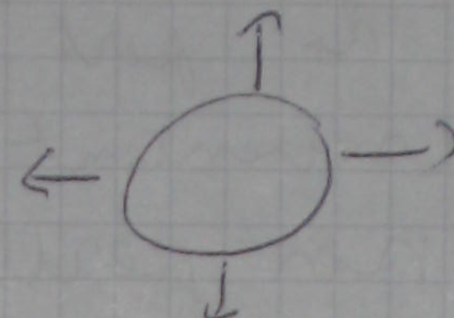
- solid angle $\Omega$ is integer multiple of $4\pi$ for any smooth configuration $\vec{\varphi}(\tau, x)$ obeying this b.c.

$S = i\theta Q$

$Q$ = top charge = $\pi^2(S^2)$



$Q=0$



$Q=1$

etc

"Can't peel an orange without splitting the skin"

$$Z = \int [d\varphi]\, e^{-S_0 + i\theta Q}$$

$$Q \in Z \Rightarrow \theta \Leftrightarrow \theta + 2\pi$$

$$\theta = 2\pi S \Leftrightarrow 0 \text{ for } S \text{ integer}$$

$$\Leftrightarrow \pi \text{ for } S \ \tfrac{1}{2}\text{-integer}$$

$\Rightarrow$ 2 different large-$S$ limits for spin chains

$\theta = 0$ case can be well-understood from perturbative $\beta$-fn, large-$N$ approx

$$\vec{\varphi} = (\varphi^1, \varphi^2, \cdots \varphi^N), \quad \vec{\varphi}^2 = 1$$

& results from integrability

$\theta = \pi$ case [$\tfrac{1}{2}$-integer $S$] is best understood from $S = \tfrac{1}{2}$ example

<u>$\theta = 0$ Case</u>

- <u>$\beta$-function</u>:

$$\frac{dg}{d\ln\Lambda} = -\frac{1}{2\pi} g^2 + \cdots \qquad (\Lambda = \text{cut-off scale})$$

$$g(\Lambda) \sim \frac{g_0}{1 - \frac{g_0}{2\pi}\ln\frac{\Lambda_0}{\Lambda}} \sim 1 \text{ at } \Lambda \sim \Lambda_0 e^{-2\pi/g_0}$$

$$\Rightarrow \Delta \sim e^{-\pi S}$$

$$\xi \sim e^{\pi S}$$

- including next order

$$\frac{dg}{d\ln\Lambda} = -\frac{g^2}{2\pi} + \frac{g^3}{4\pi} + \cdots$$

$$\xi \sim \frac{1}{S} e^{\pi S} \left[1 + O\left(\tfrac{1}{S}\right)\right]$$ - is it a true correlation length?

- agrees quite well with DMRG

$\xi(1) = 6.1, \quad \xi(2) = 49$ - even for $S = 1, 2$ (!!)

- massive nature of model confirmed by large-n limit

$$S = \int [d\vec{\varphi}][d\lambda]\, e^{-\frac{N}{2g}\int \left[(\partial\vec{\varphi})^2 + i\lambda(\vec{\varphi}^2 - 1)\right]}$$

(rescale coupling)

$$= \int [d\lambda]\, e^{-S_{eff}(\lambda)}$$

$$S_{eff}(\lambda) = -\frac{iN}{2g}\int \lambda + \frac{N}{2}\operatorname{tr}\ln(-\partial^2 + i\lambda)$$

- common factor of N => integral dominated by saddle pt where $\frac{\delta S_{eff}}{\delta\lambda} = 0$

- occurs for a purely imaginary, constant $\lambda$

$$\frac{1}{g} = \int \frac{d^2K}{(2\pi)^2} \frac{1}{K^2 + i\lambda}, \quad i\lambda = m^2$$

$$\sim \frac{1}{2\pi} \ln\frac{\Lambda}{m}$$

$$m \sim \Lambda e^{-2\pi/g}$$

- to lowest order in $1/N$

$$\langle \varphi^a \varphi^b \rangle = \frac{\delta^{ab}}{-\partial^2 + m^2}$$ - free boson

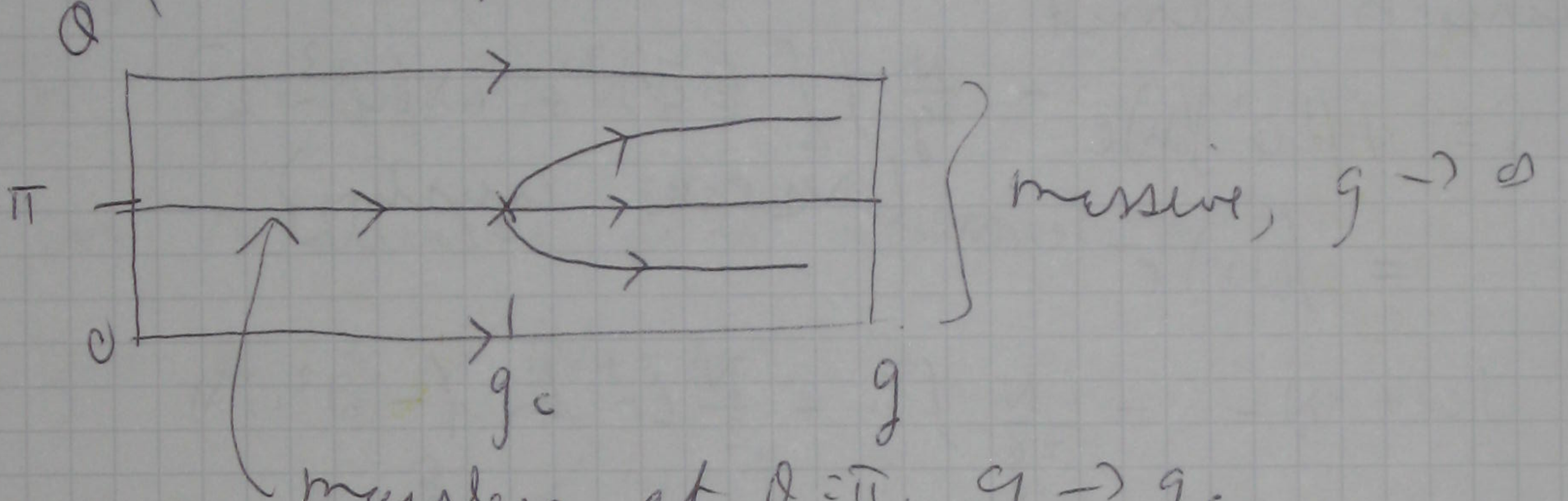
up to $\frac{1}{N}$ corrections

- likewise

$$\langle l^a \, l^b \rangle \sim \langle (\vec{\varphi} \times \partial_0 \vec{\varphi})^a \, (\vec{\varphi} \times \partial_0 \vec{\varphi})^b \rangle$$

$$\sim w^2 \langle \varphi \varphi \rangle \langle \varphi \varphi \rangle$$ - factorized

- N-tuple of massive bosons with weak repulsive interactions [no bound states] at large-N.
- on the other hand, as we will see from $S = \frac{1}{2}$ example, $S = \frac{1}{2}$-integer, $\theta = \pi$ model is massless




- this prediction about NLσM, which first came from spin chains, has been verified by direct Euclidean space Monte Carlo (~~Wiese~~ Wiese) & a proposed massless S-matrix ($\mathrm{Zam}^2$)

<u>Integrability of NLσM ($\theta = 0$)</u>

E-L eqn $\quad \partial^2 \vec{\varphi} = \lambda \vec{\varphi}$

let $\quad J = [(\partial_+ \vec{\varphi}) \cdot (\partial_+ \vec{\varphi})]^n, \quad \partial_\pm = \partial_0 \pm \partial_1$

$$\partial_- J = 2n \, (\partial_+ \vec{\varphi} \cdot \partial_+ \vec{\varphi})^{n-1} \, (\partial_+ \vec{\varphi}) \cdot (\partial_+ \partial_- \vec{\varphi})$$

$$= \quad \text{''} \quad \lambda \, (\partial_+ \vec{\varphi}) \cdot \vec{\varphi}$$

$$\propto \quad \partial_+ (\vec{\varphi}^{\,2}) = 0$$

- ~~Dirac~~ anomalies?
- exact S-matrix proposed - checked at large-N
- describes scattering of triplet of bosons, with no bound states
- Many low energy properties of spin chains can be determined from exact results on NLσM or even from free massless boson (large-N) approx
- eg - susceptibility

$$\chi = \frac{1}{T} \langle (S_T^z)^2 \rangle$$

- in relativistic free boson approx

$$\chi = \frac{1}{T} \langle (N_+ - N_-)^2 \rangle$$

$$= \frac{2}{T} \langle (N - \bar{N})^2 \rangle$$

$$= \frac{2}{T} \int \frac{d^3k}{(2\pi)^3} \frac{e^{+\sqrt{\Delta^2+k^2}/T}}{\left(e^{\sqrt{\Delta^2+k^2}/T} - 1\right)^2}$$

for $T \ll \Delta$, $\sqrt{\Delta^2 + k^2} = \Delta + \frac{k^2}{2\Delta}$

$$\chi = \frac{2}{T} \int \frac{d^3k}{(2\pi)^3} e^{-\frac{\Delta}{T} - \frac{k^2}{2\Delta T}}$$

$$= \frac{2e^{-\Delta/T}}{v} \sqrt{\frac{\Delta}{2\pi T}} \quad \text{[reinserting } v\text{]}$$

- expect this to be asymptotically exact because
1) only depends on dispersion relation at $k \to 0$ ~~(Rest)~~ $E = \Delta + \frac{k^2 v^2}{2\Delta}, \ldots$ defines $\Delta, v$

2) at $T \ll \Delta$, density of bosons is small
- interaction effects $O[e^{-2\Delta/T}]$

- exact NL$\sigma$M susceptibility is quite different (except at $T \ll \Delta$)
- agrees much better with $S=1$ chain numerical results

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$E = \sqrt{\Delta^2 + (k-\pi)^2}$ works quite well for

$$|\pi - k| \leq \pi/5 \text{ or } \pi/10$$

- neutron scattering X-sec ($T=0$)

$$\propto \int_{-\infty}^{\infty} dt \sum_j e^{i(\omega t - kj)} \langle 0| S^a_j(t) S^a_0(0) |0\rangle \sim S(k,\omega)$$

for $k \approx \pi$

$$S(\pi + k, \omega) \sim \langle \phi \phi \rangle (k, \omega)$$

for $k \approx 0$

$$S(k,\omega) \sim \langle l \, l \rangle (k,\omega)$$

- for intermediate $k$, field theory fails
- in free boson approx

$$S(\pi + k) \sim \operatorname{Im} \frac{1}{\omega^2 - k^2 \cancel{+i\epsilon} - \Delta^2 + i\epsilon}$$

$$\propto \frac{\delta(\omega - \sqrt{\Delta^2 + k^2})}{\sqrt{\Delta^2 + k^2}}$$

- interactions produce additional spectral weight at higher $\omega$ but, in NL$\sigma$M
1) $\delta$-fn pole with this $k$-dependence $\propto Z$
2) - no additional weight at $k=0$ below $3\Delta$
- since no bound states

- $2^{nd}$ feature appears to be exact
- near $k=0$:

$$\mathrm{Im}\left[\langle (\vec{\phi}\times\partial_0\vec{\phi})\,(\vec{\phi}\times\partial_0\vec{\phi})\rangle\right]$$

- purely a 2-particle continuum in free boson approx $\omega \geq 2\sqrt{\Delta^2+(k/2)^2}$ $\quad \xrightarrow[k/2]{} \;\; \xrightarrow[k/2]{}$

$$\propto k^2\,\theta\left[\omega - 2\sqrt{\Delta^2+(k/2)^2}\right]$$

- again exact at $k\to 0$

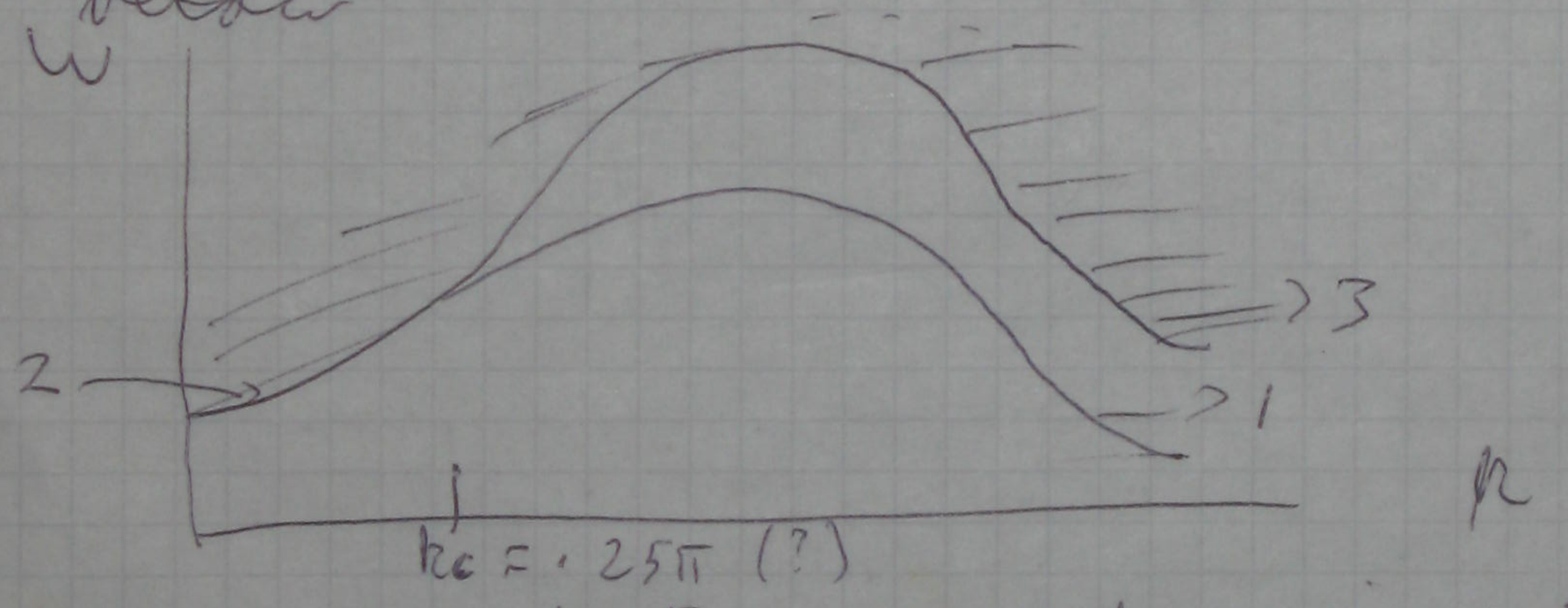
$k^2$ factor follows from conservation of $S^z_T$

$$S^z_T|0\rangle = 0$$

$$\Rightarrow \sum_j \langle 0|S^z_j S^z_0|0\rangle e^{ikj}$$

$$\to -k^2 \sum_j j^2 \langle 0|S^z_j S^z_0|0\rangle$$

- ~~no further spectral~~ - no spectral wt below $2\Delta$ again a consequence of no bnd states
- ~~expect no further spectral wt at $k=0$ below~~




NLσM prediction near $k=0$ is better than free boson but only "semi-quantitative" - better for $S \geq 2$ (?)

using NLσM prediction near $k=\pi$
[e cutting off max k of each boson)
gives ~ 2% spectral weight in 3 bosons
- again agrees well with DMRG

- to compare to experiment:
- anisotropy
- inter-chain coupling
- finite T

- can easily include anisotropy in free boson approx:

$$\cancel{\Delta \to} \Delta_x, \Delta_y, \Delta_z$$

$$\Rightarrow \mathcal{L} \to \frac{1}{2}(\partial_\mu \vec{\varphi})^2 - \frac{1}{2}\sum \Delta_i^2 \varphi_i^2$$

$$E_i = \sqrt{\Delta_i^2 + v^2 k^2} \qquad (\text{or } v \to v_i)$$

- again expect single particle poles at $k=\pi$
- seen in NENP at low T
($J'/J \sim .0004$)
- $S(k,\omega)$ is difficult to measure at small k
- too small
- apparently not measured below $k_c$
- $S(k,\omega)$ at higher $\omega$: detailed measurements in $CsNiCl_3$ ($J'/J \simeq .02$) agree poorly with theory - presumably due to inter-chain coupling e finite T (necessary to be in "1D phase")