Contents

1. Introduction

Other lectures in this series discuss the application of conformal field theory to strings and to two-dimensional *classical* critical phenomena (i.e. $(2+0)$ dimensions). These lectures will discuss its application to one-dimensional *quantum* critical phenomena (i.e. (1+1) dimensions).

More generally, field theory methods have been applied with great success to a large variety of classical critical phenomena problems in various dimensions, especially during the 1970's after the development of the renormalization group. By now, most of these problems have been understood in great detail. The frontier area presently appears to be quantum critical phenomena. A large variety of interesting many body quantum effects have been studied experimentally in recent years with the advent of improved materials preparation, higher magnetic fields and lower temperatures. These include the fractional quantum Hall effect, heavy fermions and high-T_c superconductors.

While the applicability of *Euclidean* invariant field theories near the critical point for classical systems, where the correlation length is much longer than the microscopic length scale, is fairly self-evident, the situation is much more subtle for quantum problems. In general the low-energy, long-wavelength behavior of a quantum condensed matter system may bear no relation to any Lorentz invariant quantum field theory, even in cases where the correlation length is very large, and the energy gap very small. A necessary (but not sufficient) condition for a Lorentz-invariant theory to apply is that the quasi-particle dispersion relation be relativistic at low energies. This is only true in special cases. To mention two examples that will be discussed in detail in the next chapter, a quantum *ferromagnet* has a low-energy dispersion relation $E \propto \mathbf{k}^2$, while a quantum *antiferromagnet* has $E = v|\mathbf{k}|$, where v, the spin-wave velocity, is typically some 10^6 times smaller than the true velocity of light. Thus only the latter can (and in fact does) have a Lorentz invariant low-energy effective field theory. More generally, condensed matter problems are formulated directly in terms of electron operators (rather than localized spin operators). Such problems do not in general appear

Lorentz invariant. Basically a Fermi surface is inconsistent with Lorentz invariance except in one dimension where the Fermi surface reduces to two Fermi points.

In these notes I will discuss some problems in quantum critical phenomena where Lorentz invariant quantum field theory techniques are useful. Some (but not all) of these techniques will involve conformal field theory. The two basic models I will be discussing are the quantum antiferromagnet and the Hubbard Model of itinerant electrons with short-range repulsion. I will begin my discussion of each model with a review of the standard mean field theories, which form the basis of the condensed matter physicists' understanding of the models and should be applicable in sufficiently high dimension. I will then focus on the $(1+1)$ dimensional case, using Lorentz invariant field theory techniques.

My discussion of antiferromagnetic chains will involve two distinct approaches, which bring in quite different quantum field theories. The first approach is based on the large-spin, semi-classical limit. In this limit there is a large characteristic length scale, of $O(e^{\pi s})$, which will justify a continuum approximation, giving the $O(3)$ non-linear σ-model. I will argue that integer and half-odd-integer-s are distinguished by the topological angle in the quantum field theory being 0 or π respectively. The second approach, which is only valid for massless spin chains (which, I shall argue, corresponds to half-integer-s), is based on fermionization and then bosonization. It works most simply for $s = 1/2$. In general, it leads to the SU(2) Wess–Zumino–Witten non-linear σ-models (WZW models). By extending this bosonization approach to arbitrary s I will eventually be able to come full circle and understand certain non-trivial relations between these two, apparently very different, quantum field theories.

Although the Hubbard Model is of considerable interest in its own right, I will use it in these lectures primarily as a tool for fermionizing quantum spin chains. The large-coupling limit of the Hubbard model, for a half-filled band, is the antiferromagnet. The Hubbard model can be straightforwardly studied in the continuum limit by bosonization.

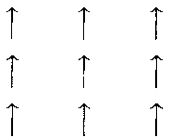
2. Spin-wave theory for the Heisenberg model [1]

The Heisenberg Hamiltonian describes the nearest neighbor interactions of localized quantum spins

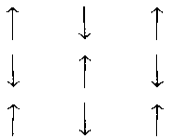
$$H = J \sum_{<ij>} \mathbf{S}_i \cdot \mathbf{S}_j.$$

(I will let $< i, j >$ denote a pair of nearest neighbor sites on some regular lattice. The sum is over each pair of n.n. sites once.) Ferromagnetism ($J < 0$) arises in atomic physics from the Coulomb repulsion of a pair of outer shell electrons on neighboring atoms. The electrons stay further apart in the parallel spin state, due to Fermi statistics. Antiferromagnetism ($J > 0$) arises from the Coulombic attraction of neighboring electrons to each other's ionic core. This attraction makes the electrons want to be closer to each other, which they achieve in the anti-parallel spin state. We will see later how this arises from the Hubbard model.

If the $\mathbf{S}_i$ were *classical* spins, i.e. fixed length classical vectors, the ground state of H would be trivially



for the ferromagnetic case or



for the antiferromagnetic case. This classical antiferromagnetic ground state is known as the Néel state. It exists for any bipartite lattice; i.e. one with two sub-lattices such that the nearest neighbors of any point in one sub-lattice are all in the other. The classical model has interesting and non-trivial behavior at finite temperature. In particular there is a phase transition from ordered to disordered states at a finite T_c. But we will be interested in the quantum case:

$$[\mathbf{S}^a, \mathbf{S}^b] = i\varepsilon^{abc}\mathbf{S}^c, \quad \mathbf{S}^2 = s(s+1).$$

(repeated indices will normally be summed in these notes). It is illuminating to rewrite H in terms of raising and lowering operators:

$$S^{\pm} = S^x \pm iS^y.$$
$$H = \sum_{<ij>} \left[\frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+) + S_i^z S_j^z\right].$$

The ordered states drawn above represent states with

$$S_i^z \mid \psi > = \pm s \mid \psi > .$$

Note that the ferromagnetic state is indeed the quantum ground state, but the Néel state is not an eigenstate since acting with H we may raise and lower a pair of neighboring spins. Much of these lectures will be concerned with the nature of the true ground states (and low energy excitations) of antiferromagnets. It is in general a rich and complex subject.

Since the Néel state is not the exact ground state for the antiferromagnetic Heisenberg model, we would like to know whether or not, and under what circumstances, it is a qualitatively correct approximation to the ground state. In particular, we would like to know whether or not long-range antiferromagnetic order:

$$\langle S^z(\mathbf{x}) S^z(\mathbf{0})\rangle = \pm m,$$

occurs. (The + or = sign occurs depending on whether the points $\mathbf{0}$ and $\mathbf{x}$ are on the same or opposite sub-lattice.) To study this question it is convenient to perturb away from a limit in which the Néel state becomes exact. This occurs for a spin-s Heisenberg antiferromagnet in the limit $s \to \infty$. In this limit, quantum spins become classical because the commutator is much smaller than the square of the spin variables:

$$\left[S^a, S^b\right] = i\varepsilon^{abc} \quad S^c = O(s) << O\left(s^2\right).$$

For large s we expect S^z to make only relatively small deviations away from $\pm s$ (the + or − sign occurs for one or the other sublattice.) A nice way of formalizing this and studying the corrections to the Néel state is to use the Holstein–Primakov transformation. I leave it as an exercise to show that the spin operators can be represented in terms of a boson, annihilation operator, "a" as:

$$S^z = s - a^\dagger a, \quad S^- = \sqrt{2s} a^\dagger \left(1 - a^\dagger a/2s\right)^{1/2}.$$

This produces the exact commutation relations and the constraint $\mathbf{S}^2 = s(s+1)$. Note that the state with maximal $S^z, S^z = s$, corresponds to the state with no bosons present. Acting on this state with S^- creates a boson and hence lowers S^z by one. The state of minimal S^z, $S^z = -s$, is the state with S^- bosons present. This state is annihilated by S^-, as it should be since $1 - a^\dagger a/2s = 0$ acting on this state. Using this representation for the Heisenberg model we will find that $a^\dagger a$ is $O(1)$ so that S^z deviates from s by an amount of $O(1) << s$.

We may develop a systematic expansion in $1/s$ by expanding the $\sqrt{}$ in the definition of S^-:

$$S^- \approx \sqrt{2s} a^\dagger.$$

In the ferromagnetic case we obtain, to quadratic order:

$$H = Js \sum_{<ij>} \left[-a_i^\dagger a_i - a_j^\dagger a_j + a_i^\dagger a_j + a_j^\dagger a_i\right],$$

or, in momentum space:

$$H = |J| Sz \sum_{\mathbf{k}} (1 - \gamma_{\mathbf{k}})\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}},$$

where z is the number of nearest neighbors and

$$\gamma_{\mathbf{k}} \equiv \sum_{\boldsymbol{\delta}} e^{i\mathbf{k}\cdot\boldsymbol{\delta}}/z = \gamma_{-\mathbf{k}}.$$

Here $\boldsymbol{\delta}$ represent the translation vectors to the neighboring sites. (We assume a Bravais lattice; i.e. all sites are equivalent.) At small $\mathbf{k}$, we have

$$E(\mathbf{k}) \to Js\mathbf{k}^2,$$

a non-relativistic dispersion relation.

For antiferromagnets we must proceed somewhat differently. We make the above expansion on one sub-lattice, the "A" sublattice, but on the other "B" sublattice, it is convenient to represent the spins in terms of a boson, b:

$$S^z = -s + b^\dagger b, \quad S^- = \sqrt{2s}\left(1 - b^\dagger b/2s\right)^{1/2} b.$$

Again the state with no bosons corresponds to the Néel state. In this case acting with S^+ creates bosons. To quadratic order the Hamiltonian is:

$$H = J \sum_{<\mathbf{x},\mathbf{y}>} \left[-s^2 + s\left(a_{\mathbf{x}}^\dagger a_{\mathbf{x}} + b_{\mathbf{y}}^\dagger b_{\mathbf{y}} + a_{\mathbf{x}} b_{\mathbf{y}} + a_{\mathbf{x}}^\dagger b_{\mathbf{y}}^\dagger\right) + \cdots\right]$$

Here we sum over nearest neighbors with $\mathbf{x}, \mathbf{y}$ in the a and b sublattice respectively. The $\cdots$ represent higher order terms in the boson operators. These terms are down by powers of s relative to the terms we kept, and hence can be treated in perturbation theory in $1/s$. To leading order we must diagonalize the quadratic Hamiltonian. We do this by going to momentum space and then diagonalizing by a Bogoliubov transformation. In momentum space:

$$H = Jsz \sum_{\mathbf{k}} \left[\gamma_{\mathbf{k}} \left(a_{\mathbf{k}} b_{\mathbf{k}} + a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger \right) + \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right) \right],$$

We see that we must make a Bogoliubov transformation mixing $a_{\mathbf{k}}$ with $b_{\mathbf{k}}$. We define new boson operators by the canonical transformation:

$$c_{\mathbf{k}} = u_{\mathbf{k}} a_{\mathbf{k}} - v_{\mathbf{k}} b_{\mathbf{k}}^\dagger, \quad d_{\mathbf{k}} = u_{\mathbf{k}} b_{\mathbf{k}} - v_{\mathbf{k}} a_{\mathbf{k}}^\dagger,$$

The correct normalization of the commutators of c with $c^\dagger$ and d with $d^\dagger$ imply:

$$|u|^2 - |v|^2 = 1.$$

Inverting the transformation and plugging into H, we find the condition that make the off-diagonal terms vanish:

$$\gamma \left(u^2 + v^2 \right) + 2uv = 0.$$

Solving for u and v we find the diagonal Hamiltonian:

$$H = Jsz \sum_{\mathbf{k}} \left[1 - \gamma_{\mathbf{k}}^2 \right]^{1/2} \left(c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + d_{\mathbf{k}}^\dagger d_{\mathbf{k}} \right).$$

The excitations created by c and d are known as spin-waves. They correspond to infinitesimal deviations of the spins away from the Néel state. The spin-wave dispersion relation is:

$$E(\mathbf{k}) = Jsz \left[1 - \gamma_{\mathbf{k}}^2 \right]^{1/2}.$$

Note that $E(\mathbf{k})$ vanishes linearly at $\mathbf{k} = 0$:

$$E(\mathbf{k}) \to u|\mathbf{k}|$$

For a d-dimensional hypercubic lattice, we have:

$$u = 2\sqrt{d} Js.$$

This is a relativistic dispersion relation, with u the effective "velocity of light". We will shortly discover a relativistic quantum field theory description of the low-energy theory. The vanishing of the spin-wave gap is a consequence of Goldstone's theorem. There are two Goldstone modes, c and d, corresponding to the breaking of SO(3) down to SO(2), rotations about the z axis. Long wavelength spin-waves correspond to states where all regions are locally in some Néel ground state but the direction of the sub-lattice magnetization vector makes long wavelength rotations. Note that the two different Goldstone modes, c and d have spin $S^z = \pm 1$, respectively. They correspond to raising S^z on one sub-lattice or lowering it on the other. This doubling of low-energy modes, compared to the ferromagnetic case is a consequence of the doubling of the unit cell in the Néel phase. The antiferromagnetic unit cell, with parallel spins at the corners, has twice the volume of the original one.

We may use the spin-wave representation to calculate the reduction in the sub-lattice magnetization due to quantum fluctuations:

$$\langle S_z(x) \rangle = s - \left\langle a^\dagger(\mathbf{x}) a(\mathbf{x}) \right\rangle = s - \sum_{\mathbf{k}} \left\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \right\rangle / V.$$

(where V is the volume measured in units where the Néel unit cell has area 1).

Inverting the Bogoliubov transformation gives:

$$a_{\mathbf{k}} = u_{\mathbf{k}} c_{\mathbf{k}} + v_{\mathbf{k}} d_{\mathbf{k}}^\dagger.$$

Since $< c_{\mathbf{k}}^\dagger c_{\mathbf{k}} > = < d_{\mathbf{k}}^\dagger d_{\mathbf{k}} > = 0$, in the ground state, we have:

$$\begin{aligned} \langle S_z(\mathbf{x}) \rangle &= s - \sum_{\mathbf{k}} v_{\mathbf{k}}^2 / V \\ &= s - \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{1}{2} \left[\left(1 - \gamma_{\mathbf{k}}^2 \right)^{-1/2} - 1 \right]. \end{aligned}$$

Essentially the quantum spins have zero-point motion, just like a quantum harmonic oscillator. Just as

$$\langle x^2 \rangle = \frac{1}{2\omega}.$$

is non-zero for a harmonic oscillator of frequency ω (and mass$= 1$) so $< s - S_z >$ is non-zero for the quantum spins. Indeed the 2 formulae are

almost the same. Note that the zero-point reduction is independent of s, to leading order, whereas the classical value is s. Thus the quantum correction is relatively small for large s. In fact we have calculated the first two terms in a power series in $1/s$ for m. We find:

$$m \approx s - .1 + O\left(\frac{1}{s}\right) \qquad (d = 3 \text{ cubic lattice})$$

$$m \approx s - .2 + O\left(\frac{1}{s}\right) \qquad (d = 2 \text{ square lattice}).$$

However, for $d = 1$ we find that the "correction" is divergent at small wave-numbers:

$$\Delta \langle S_z(\mathbf{x}) \rangle \approx - \int \frac{\mathrm{d}k}{2\pi} \frac{1}{2k} = -\infty.$$

This indicates that the Néel state is destabilized by quantum fluctuations in $d = 1$, no matter how large s is. This is a familiar result to a quantum field theorist: Coleman's theorem [2] states that spontaneous breaking of continuous symmetries is impossible in $(1 + 1)$ dimensions, due to the infra-red divergences connected with the Goldstone bosons.

Note that for small values of s, in particular, $s = 1/2$, the quantum correction is quite significant also for $d = 2$: m is reduced form .5 to .3. It is not obvious that the $O(1/s)$ terms are negligible. One could imagine that quantum fluctuations might destroy the Néel order in this case as well. The tendency towards destruction of Néel order by quantum fluctuations is enhanced by effects like frustrating next nearest neighbor couplings:

$$H \to J_1 \sum_{\mathrm{nn}} \mathbf{S_x} \cdot \mathbf{S_y} + J_2 \sum_{\mathrm{nnn}} \mathbf{S_x} \cdot \mathbf{S_y}.$$

If J_2 and J_1 are both positive then next nearest neighbors as well as nearest neighbors want to be anti-parallel, classically. Since this is not possible, there is a tendency to destroy the ordered state.

We may repeat the approximate diagonalization of H in this case. The reduction in the sub-lattice magnetization due to quantum fluctuations is increased for positive J_2 to:

$$\langle S_z(\mathbf{x}) \rangle = s - \int \frac{\mathrm{d}^d\mathbf{k}}{2\pi^d} \frac{1}{2} \left[\beta_{\mathbf{k}} (\beta_{\mathbf{k}}^2 - \gamma_{\mathbf{k}}^2)^{-1/2} - 1 \right],$$

where

$$\beta_{\mathbf{k}} \equiv 1 - \frac{J_2 z_2}{J_1 z_1} \left[1 - \frac{1}{z_2} \sum_{\delta} \mathrm{e}^{i\mathbf{k}\cdot\delta} \right],$$

where z_2 is the number of next nearest neighbors and the δ_2 are the vectors between next nearest neighbors. For $J_2 > 0$, $\beta_{\mathbf{k}} < 1$, so the quantum zero-point motion is *increased*.

Thus we should not take Néel order for granted, even for the pure nearest neighbor Heisenberg model, at $s = 1/2$. However, there is a rigorous proof of Néel order for a large number of cases [3,4]. The proof works for a nearest neighbor Heisenberg model on a square lattice ($d = 2$) for $s \geq 1$ and on a cubic lattice ($d = 3$) for all s. The theoretical and experimental evidence collected over the last six months seems quite strongly suggestive of Néel order for the $d = 2$, $s = 1/2$ case as well. Rigorous examples [4] of Hamiltonians with disordered ground states also exist in two dimensions. They contain higher powers of the nearest neighbor product $(\mathbf{S}_i \cdot \mathbf{S}_j)^2$. The mapping onto a quantum field theory may give a unified description of the finite temperature ordering transition for a classical magnet and the zero temperature transition for a quantum antiferromagnet, as a function of parameters in the Hamiltonian.

3. Large-s mapping onto the $O(3)$ non-linear σ-models [5]

We now will try to derive the low-energy, continuum limit of quantum magnets. The basic idea is to keep only the long wavelength, low-energy modes. Formally, we may "integrate out" the high-energy modes to obtain an effective Hamiltonian. This treatment can be justified in the large$-s$ limit, although many of the conclusions should remain true for arbitrary s.

In the ferromagnetic case, we may regard $\mathbf{S}$ as a classical vector for large s, $\mathbf{S} = s\boldsymbol{\varphi}$, with $\boldsymbol{\varphi}^2 = 1$. At long wavelength we obtain:

$$H \approx Js^2 \int \mathrm{d}^d\mathbf{k} (\nabla\boldsymbol{\varphi})^2.$$

This is just a classical system. We obtain the quadratic dispersion relation, $E \propto \mathbf{k}^2$.

The antiferromagnetic case is considerably less trivial due to the fact that there are important low energy effects at wave vectors near $\mathbf{0}$ and also near the ordering wave vector, $\mathbf{k}_0$. It was this doubling of the unit cell which led to a linear dispersion relation at low energy. We wish to keep the Fourier modes of $\mathbf{S}_i$ near these 2 values of $\mathbf{k}$. Thus we write

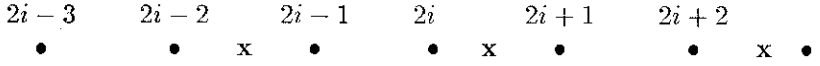
$$\mathbf{S}_i \approx \pm s\boldsymbol{\varphi}_i + \mathbf{l}_i,$$

where φ and $\mathbf{l}$ are slowly varying on the scale of the lattice. In the continuum limit, φ will become the field of the $O(3)$ non-linear σ-model and $\mathbf{l}$ will become the generator of rotations,

$$\mathbf{l} \approx \varphi \times \frac{\mathrm{d}\varphi}{\mathrm{d}t}.$$

These obey the constraints, $\varphi^2 = 1$, $\varphi \cdot \mathbf{l} = 0$. We expect the $\mathbf{k} \approx 0$ component to be $O(1)$ and the $\mathbf{k} \approx \mathbf{k}_0$ component to be $O(s)$, based on the classical or Holstein–Primakoff analysis. To set this up in a careful, quantum mechanical way, it is convenient to define the φ and $\mathbf{l}$ variables as linear combinations of a few nearby sites. I only consider the one-dimensional case here.

We make the arbitrary choice of combining each spin on an even site, $2i$ with the spin to its right, at $(2i+1)$.



$$\varphi\left(2i+\tfrac{1}{2}\right) = \left[\mathbf{S}_{2i+1} - \mathbf{S}_{2i}\right], \quad \mathbf{l}\left(2i+\tfrac{1}{2}\right) = \left[\mathbf{S}_{2i} - \mathbf{S}_{2i+1}\right]/2. \qquad (1)$$

This choice breaks the symmetries of translation by one site and of parity, reflection about a site. We shall see that these symmetries are restored in the continuum limit. The advantage of this approach is that the commutators are simple:

$$\begin{aligned}
\left[l^a(x),\ l^b(y)\right] &= i\varepsilon^{abc} l^c \delta(x-y) \\
\left[l^a(x),\ \varphi^b(y)\right] &= i\varepsilon^{abc} l^c \varphi(x-y) \\
\left[\varphi^a(x),\ \varphi^b(y)\right] &= i\varepsilon^{abc} l^c \delta(x-y)/s^2 \to 0.
\end{aligned}$$

Here the Dirac delta function is defined from the continuum limit of the Kronecker δ-function as

$$\delta(x-y) = \lim \delta_{x,y}/a,$$

where, in this case, the lattice spacing between the points where the fields φ and $\mathbf{l}$ are defined, is $a = 2$. The first 2 commutation relations are those of the rotation generator and field in the σ-model. The last commutator only becomes that of the field theory as $s \to \infty$. This is the fundamental approximation of the large-s limit. As defined by (1), φ and $\mathbf{l}$ obey the constraints: $\varphi \cdot \mathbf{l} = 0$, and

$$\varphi^2 = 1 + 1/s - \mathbf{l}^2/s^2 \to 1.$$

We now assume that φ and $\mathbf{l}$ are slowly varying on the lattice scale and calculate the Hamiltonian in a gradient expansion. We keep terms up to $O(\varphi'^2)$ and $O(\mathbf{l}^2)$, only since, $\mathbf{l}$ effectively contains a time derivative. Somewhat surprisingly, we will pick up a term *linear* in (d/dx) which appears to break parity. This is a consequence of the parity breaking method of passing to the continuum limit introduced above. Using

$$\begin{aligned}
\mathbf{S}_{2i} \cdot \mathbf{S}_{2i+1} =& 2\mathbf{l}\left(2i+\frac{1}{2}\right)^2 + \text{constant}, \\
\mathbf{S}_{2i} \cdot \mathbf{S}_{2i-1} =& -s^2 \varphi\left(2i+\frac{1}{2}\right) \cdot \varphi\left(2\left(2i-\frac{3}{2}\right)\right. \\
&+ s\left[\mathbf{l}\left(2i-\frac{3}{2}\right) \cdot \varphi\left(2i+\frac{1}{2}\right) - \varphi\left(2i-\frac{3}{2}\right) \cdot \mathbf{l}\left(2i+\frac{1}{2}\right)\right] \\
&+ \mathbf{l}\left(2i+\frac{1}{2}\right) \cdot 1\left(2i-\frac{3}{2}\right) \\
\approx& 2s^2\varphi'^2 - s(\mathbf{l} \cdot \varphi' + \varphi' \cdot \mathbf{l}) + 2\mathbf{l}^2 + \text{constant}.
\end{aligned}$$

The Hamiltonian then becomes (dropping a constant):

$$H = J \int \frac{\mathrm{d}x}{2} \left[4\mathbf{l}^2 + 2s^2\varphi'^2 - s(\mathbf{l} \cdot \varphi' + \varphi' \cdot \mathbf{l})\right].$$

We rewrite this as:

$$H = \frac{v}{2}\left\{ g^2 \left[\mathbf{l} - \frac{\theta}{4\pi}\varphi'\right]^2 + \frac{\varphi'^2}{g^2}\right\}. \qquad (2)$$

With velocity, coupling constant and topological angle:

$$v = 2Js, \quad g = \frac{2}{s}, \quad \theta = 2\pi s.$$

To understand better what this Hamiltonian represents, we observe that it follows from the Lagrangian:

$$L = \frac{1}{2g}\partial_\mu \varphi \partial^\mu \varphi + \frac{\theta}{8\pi}\varepsilon^{\mu\nu}\varphi \cdot (\partial_\mu \varphi \times \partial_\nu \varphi),$$

with the constraint, $\varphi^2 = 1$. (We set $v = 1$ for this discussion.) The simplest way of handling the constraint is to introduce co-ordinates for the sphere:

$$\begin{aligned}
\varphi =& (\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ \cos\alpha). \\
L =& \frac{1}{2g}\left[(\partial_\mu\alpha)^2 + \sin^2\alpha\,(\partial_\mu\beta)^2\right] + \frac{\theta}{8\pi}\sin\alpha\varepsilon^{\mu\nu}\partial_\mu\alpha\partial_\nu\beta.
\end{aligned}$$

We now see that the θ-term is a total derivative. On the compactified Euclidean space (i.e. with $\varphi(\mathbf{x}) \to$ constant at $|\mathbf{x}| \to \infty$) both φ and $\mathbf{x}$ can be represented by points on a sphere, S^2. The integral

$$\frac{1}{8\pi}\int \mathrm{d}^2\mathbf{x}\varepsilon^{\mu\nu}\varphi\cdot(\partial_\mu\varphi\times\partial_\nu\varphi),$$

measures the winding number of the sphere onto the sphere, i.e. $\Pi^2(S^2)$. This is an integer for any smooth (finite action) field configuration. Since the θ-term is a total derivative it has no effect on the classical equations of motion, and no effect in perturbation theory. It does however, lead to a change in the Hamiltonian. The conjugate momenta are:

$$\Pi_\alpha = \frac{\dot\alpha}{g} + \frac{\theta}{4\pi}\beta'\sin\alpha$$
$$\Pi_\beta = \frac{\dot\beta\sin^2\alpha}{g} - \frac{\theta}{4\pi}\alpha'\sin\alpha,$$

and the Hamiltonian density is

$$H = \frac{g}{2}\left\{\left[\Pi_\alpha - \frac{\theta}{4\pi}\beta'\sin\alpha\right]^2 + \left[\Pi_\beta + \frac{\theta}{4\pi}\alpha'\sin\alpha\right]^2/\sin^2\alpha\right\} + \frac{1}{2}\left[\alpha'^2+\beta'^2\sin^2\alpha\right].$$

Note that the effect of the topological term is to redefine the conjugate momenta by a canonical transformation:

$$\Pi_i \to \exp\left[i\frac{\theta}{4\pi}\int \mathrm{d}x\,\beta'\cos\alpha\right]\Pi_i \to \exp\left[-i\frac{\theta}{4\pi}\int \mathrm{d}x\,\beta'\cos\alpha\right].$$

Thus the θ-dependence of the Hamiltonian may be removed by making this transformation. However, it is expected that the Hilbert Space breaks up into decoupled sectors labelled by θ. Making the canonical transformation on the *states* maps between the different sectors, or θ-vacua.

Finally we may rewrite this Hamiltonian in a co-ordinate independent way by introducing the generator of rotations:

$$\mathbf{l} \equiv \varphi\times\left[\frac{\partial\varphi}{\partial\alpha}\Pi_\alpha + \frac{1}{\sin^2\alpha}\frac{\partial\varphi}{\partial\beta}\Pi_\beta\right].$$

The Hamiltonian then takes the form of eq. (2).

4. Behavior of the $O(3)$ non-linear σ-model: hints from the renormalization group, large-n limits and exact S-Matrix

We now comment on the implications of Hamiltonian (2) for the quantum spin chain. First note that a semi-classical analysis of (2) reproduces the results of lowest order (in $1/s$) spin-wave theory. Semi-classically we assume spontaneous symmetry breaking and write:

$$\varphi = \left(\varphi_1, \varphi_2, \sqrt{1-\varphi_1\varphi_1-\varphi_2\varphi_2}\right) \approx (\varphi_1,\varphi_2,1).$$

We then obtain the approximate free Lagrangian density:

$$L \approx \frac{1}{2g}\sum_{i=1}^{2}(\partial_\mu\varphi_i)^2.$$

Thus we obtain 2 free massless Goldstone bosons. This is the same result obtained using spin-wave theory. We also obtain the same velocity parameter, $v = 2Js$. The sub-lattice magnetization is given by

$$s\langle\varphi_3\rangle \approx s\left[1-\left\langle\sum_{i=1}^{2}\varphi_i^2\right\rangle\right] = s\left[1-2g\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\mathbf{k}^2\right].$$

We obtain the same infra-red divergence as before, telling us that the symmetry is in fact *not* spontaneously broken, since we are in $(1+1)$ dimensions.

We apply the renormalization group to obtain a deeper understanding of this infra-red divergence. The effective coupling at length scale L behaves as:

$$\frac{\mathrm{d}g}{\mathrm{d}\ln L} = \frac{g^2}{2\pi}.$$

The effective coupling grows large at large length scales;

$$g(L) \approx g_0/\left[1-\frac{g_0}{2\pi}\ln L\right].$$

Initially the bare coupling at the lattice scale is $g_0 = 2/s$. $g(L)$ becomes $O(1)$ at a length scales

$$\xi \approx \mathrm{e}^{2\pi/g_0} = \mathrm{e}^{\pi s}.$$

This implies that for large s the spin chain has short-range order, but the spins make very long wavelength, $O(\xi)$, rotations which destroy the order.

We normally expect that such a field theory develops a mass gap due to the infrared fluctuations, with a mass, $m \approx \xi^{-1}$. This result can be seen, for example in the large-n limit of the $O(n)$ σ-model [5]:

This is defined by the Lagrangian:

$$L = \frac{n}{2g}(\partial_\mu \boldsymbol{\varphi})^2, \quad \boldsymbol{\varphi}^2 = 1,$$

with $\boldsymbol{\varphi}$ an n-component vector. It is convenient to enforce the constraint using the Fourier representation of the δ -function:

$$L = \frac{n}{2g}\left[(\partial_\mu \boldsymbol{\varphi}) + i\lambda\left(\boldsymbol{\varphi}^2 - 1\right)\right].$$

We now integrate out $\boldsymbol{\varphi}$, which is no longer constrained, obtaining an effective action for λ:

$$S^{\text{eff}}(\lambda) = \frac{n}{2}\left[-\int \mathrm{d}^2\mathrm{x}\frac{i\lambda}{g} + \mathrm{tr}\ln\left(-\partial^2 + i\lambda\right)\right].$$

Because there is a factor of n in S^{eff} we may ignore the fluctuations of λ and evaluate it at the lowest action saddle point. This is obtained by deforming the functional integration contour of λ into the complex plane. A saddle point is found at constant, imaginary λ. Defining

$$i\lambda_0 = m^2 \text{ (the saddle} - \text{point value)},$$

we have the saddle-point equation:

$$\frac{1}{g} = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\frac{1}{\mathbf{k}^2 + m^2} = \frac{1}{2\pi}\ln\frac{\Lambda}{m},$$

where we have introduced an ultra-violet cut-off Λ. This equation determines the mass parameter, m, in terms of the cut-off and bare coupling, g:

$$m = \Lambda \mathrm{e}^{-2\pi/g}.$$

To lowest order in $1/n$, the $\boldsymbol{\varphi}$ Green's functions are just those of n free bosons of mass, m. Thus m defines the physical mass scale of the theory,

and we may define the renormalized coupling by the variation of g with Λ such that m stays fixed.

This picture of the $O(3)$ σ-model is reinforced by the exact S-matrix result [7]. An exact S-matrix has been found obeying all the requirements of a quantum field theory S-matrix (Lorentz invariance, unitarity, etc.) in which the only single particle states are a triplet of massive bosons. This is believed to be the S-matrix of the $O(3)$ σ-model, a conjecture which can be checked in perturbation theory.

Another type of evidence about the behavior of the $O(3)$ σ-model is based on the Euclidean lattice version of the theory:

$$S = -\frac{1}{T}\sum_{<i,j>} \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j.$$

This is simply the two-dimensional *classical* Heisenberg ferromagnet (or anti-ferromagnet; it makes no difference at the classical level). Monte Carlo simulations (and presumably experiments on magnetic systems) indicate that the system is in a disordered phase with a finite correlation length for all T. The renormalized field theory is obtained by taking $T \to 0$ as the lattice spacing goes to zero.

However, so far we have not taken into account the topological term. It has no effect in perturbation theory. However, it is expected to produce important non-perturbative effects. Note that θ should be correctly thought of an *angle* since the Euclidean action is

$$S = S_0 + i\theta Q,$$

where Q is the topological charge, an integer. Thus e^{-S} is periodic in θ. We found above that $\theta = 2\pi s$, i.e. $\theta = 0$ or π for s an integer or a half-integer. This suggests that there may be qualitative differences between the 2 cases, even for large s.

It is also instructive to think about parity. The topological term is linear is $\mathrm{d}/\mathrm{d}x$ so it is odd under parity. However, at $\theta = \pi$ parity is restored because $\mathrm{e}^{i\pi Q} = \mathrm{e}^{-i\pi Q}$ for Q integer. We commented above that the way we passed to the continuum limit of the spin chain explicitly broke parity. In particular if we had paired the sites in the opposite way we would have changed the sign of θ. But since we always obtain $\theta = 0$ or π, we see that indeed parity is present in the continuum limit. It may, however, be *spontaneously* broken.

We now see how to obtain a quantum spin chain corresponding to an arbitrary value of θ; we just need to explicitly break parity. This can be done by adding an alternating interaction.

$$H \to J \sum_i \left[1 + \gamma(-1)^i\right] \mathbf{S}_i \cdot \mathbf{S}_{i+1}.$$

Repeating the above calculations we find:

$$\theta = 2\pi s(1+\gamma), \quad g = \frac{2}{s\,(1-\gamma^2)^{1/2}}.$$

Note that at $\gamma = 1$ the chain decouples into pairs of sites. This corresponds to the strong-coupling limit, $g \to \infty$. We may also vary the strength of g without changing θ by adding a second nearest neighbor interaction:

$$H = \sum_i \left\{ J\left[1 + \gamma(-1)^i\right] \mathbf{S}_i \cdot \mathbf{S}_{i+1} + J_2 \mathbf{S}_i \cdot \mathbf{S}_{i+2} \right\}.$$

We now find:

$$\theta = 2\pi s(1+\gamma), \quad g = \frac{2}{s\,(1-\gamma^2+4J_2/J)^{1/2}}.$$

We see that a positive second nearest neighbor coupling, $J_2 > 0$, which tends to frustrate the Néel state, increases the bare coupling; i.e. it makes the chain disorder on smaller scales.

The behavior of the $O(3)$ σ-model with a topological term is much less well understood than the $\theta = 0$ model. Indeed I will try to elucidate the general behavior by first understanding spin chains.

One might again look for a large-n solution. However, since $\Pi^2(S^n) = 1$ for $n \geq 4$, we cannot use the $O(n)$ model. An alternative is to use the equivalence of $O(3)$ with SU(2) and consider instead SU(n) generalizations. This gives the CP^n models. To pass to the CP^1 formulation of the $O(3)$ model, we use the spinor representation of a unit vector:

$$\boldsymbol{\varphi} = z^* \boldsymbol{\sigma} z, \quad \text{with } |z|^2 = 1$$

(Here z_α is a complex spinor.) The Lagrangian becomes:

$$L = \frac{1}{g}\left[|\partial_\mu z|^2 + (z^* \partial_\mu z)^2\right] + \frac{\theta}{2\pi} \partial_\mu \varepsilon^{\mu\nu} (z^* \partial_\nu z). \tag{3}$$

This can be generalized to the SU(n) case by letting the unit vector, z_α have n complex components.

Much insight into this model can be gained by rewriting it in terms of an Abelian gauge field:

$$A_\mu = i z^* \partial_\mu z. \tag{4}$$

We may eliminate the $(z^* \partial_\mu z)^2$ in L by introducing A_μ:

$$L = \frac{1}{g} |(\partial_\mu + i A_\mu) z|^2 - i \frac{\theta}{2\pi} \partial_\mu \varepsilon^{\mu\nu} A_\nu.$$

Since $|z|^2 = 1$, the term in L quadratic in A is simply $(2/g)A_\mu^2$, and hence the functional integral over A_μ is Gaussian and can be done exactly, yielding Eqs.(3) and (4). Written in this form the theory looks rather like scalar QED in $(1+1)$ dimensions, except for the constraint on z and the absence of an $F_{\mu\nu}^2$ term. We can make the resemblance closer by reintroducing a Fourier transform of the δ -function:

$$L = \frac{1}{g}\left[|(\partial_\mu + i A_\mu) z|^2 + i\lambda\left(|z|^2 - 1\right)\right] - i \frac{\theta}{2\pi} \partial_\mu \varepsilon^{\mu\nu} A_\nu.$$

We may integrate out the z-fields and obtain the effective action for A_μ and λ:

$$S^{\text{eff}} = n\left\{ -\frac{1}{g} \int \mathrm{d}^2\mathbf{x}\, i\lambda + \mathrm{tr} \ln\left[-(\partial_\mu + i A_\mu)^2 + i\lambda \right] \right\} - i \frac{\theta}{2\pi} \partial_\mu \varepsilon^{\mu\nu} A_\nu.$$

We find a saddle-point with $A_\mu = 0$ and $i\lambda = m^2$ as before:

$$m = \Lambda \mathrm{e}^{-2\pi/g}.$$

Expanding around the saddle-point we pick up a term:

$$\delta L = \frac{1}{4e^2} F_{\mu\nu}^2,$$

with an effective gauge coupling:

$$\mathrm{e}^2 = 12\pi^2 m^2 / n.$$

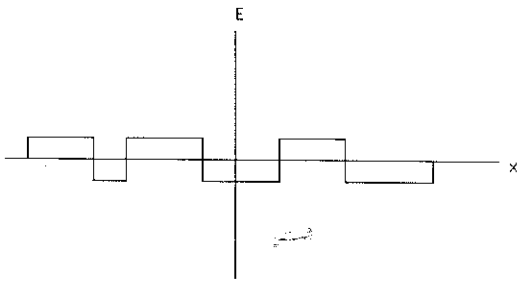
If we calculate Green's functions of the z-bosons they become those of n complex bosons of mass m which are free except for the very weak,

$O(1/n)$ abelian gauge interaction [8,9]. Weakly coupled scalar QED can be studied semi-classically, i.e. we use the non-relativistic quark model [10]. The Coulomb potential is linear in 1 space dimension leading to confinement. We obtain a spectrum of quark-antiquark bound-states from solving this two-particle quantum mechanics problem. These can transform under the singlet or adjoint representation of the SU(n) flavour group. It turns out that the lowest of these transforms under the adjoint representation. As n is decreased, the potential becomes steeper and the number of bound states decreases. It is plausible [9] that at $n \to 2$, there remains only a single triplet bound state, which can then be identified with the large-n $O(n)$ or exact S-matrix result.

So far we have been discussing the case $\theta = 0$. But the motivation for studying the CP^n models was the possibility of having $\theta \neq 0$. The effects of θ are easily understood using the non-relativistic quark model [10]. The θ -term, linear in the electric field can be interpreted as corresponding to a uniform external electric field. Equivalently, we have external charges, $\pm(\theta/2\pi)e$ at opposite ends of the one-dimensional universe, which produce the external field. Note that the solution of the 1-dimensional Poisson equation:

$$\frac{\mathrm{d}E}{\mathrm{d}x} = -\sum_i q_i \delta(x_i)$$

gives an electric field that jumps by $-q_i$ as each charge is passed.



If the external charge gets too big, dynamical e^+e^- pairs will be nucleated from the vacuum which go off to $\pm\infty$ to screen the external field. As we increase θ from zero, the nucleation first occurs at $\theta = \pi$. Sending one pair to ∞ changes the external charge at $+\infty$ from $e/2$ to $-e/2$. This is a first order phase transition. The masses of all particles are finite at the transition. Actually, for any $\theta \neq \pi$, the spectrum consists only of bound states. At $\theta = \pi$ there is no long-range force between an electron and positron provided that the electron is to the left of the positron. Thus the mass gap changes from approximately $2m$ to m at $\theta = \pi$. However, it remains finite.

While I believe that these large-n results are qualitatively correct for $n = 2$ away from $\theta = \pi$, I think they are misleading close to the transition point. While the transition is indeed first order for $n \geq 3$, I will argue that it is likely second order for $n = 2$, corresponding to the $O(3)$ σ-model.

In any event the conclusion seems clear for integer-s. There should be a finite mass gap to the triplet of bosons. This prediction of Haldane came as a big surprise to many members of the condensed matter community. Some evidence for a gap has been seem from neutron scattering experiments [11] in the quasi-one-dimensional spin-1 antiferromagnet $CsNiCl_3$. The energy-momentum relation has been measured and a gap does appear, unlike in spin$-1/2$ systems like $CuCl_2 \cdot 2(NC_5H_5)$. There are however, other, more mundane possible explanations of the observed gap, which must be ruled out. Furthermore, the effects of inter-chain coupling are large and complicate the analysis of the system.

Certain features of this derivation of the spin chain σ-model correspondence are a bit unsatisfying. We found it necessary to break parity at an intermediate stage in passing to the continuum limit. Because of the approximate commutation relations, it is not obvious how to calculate higher order corrections in $1/s$. Also, it is not clear from the derivation that corrections to θ of higher order in $1/s$ are zero. They in fact must be, since parity conserving Hamiltonians necessarily have $\theta = 0$ or π. (Actually this leaves open the possibility that $\theta = 2\pi(s + 1/2)$, but our later discussion will make it clear that this cannot be the correct result.) The path integral approach may provide an alternative derivation which remedies these defects [12].

5. The Lieb–Shultz–Mattis theorem

We saw above that there is a difference between integer and half-integer spin; in the large-s limit they map onto different field theories- with $\theta = 0$ or π respectively. While the behavior of the $\theta = 0$ theory seems to be very well understood, the $\theta = \pi$ case is much less so. Indeed, I will use this mapping as much to learn about the field theory from the spin chains, as the reverse. Our first clue about the behavior of half-integer-s isotropic spin chains comes from a rigorous theorem, first proved for $s = 1/2$ by Lieb, Shultz and Mattis [13] and then extended to arbitrary

half-integer-s in [14]. It is of special significance that the theorem only holds for half-integer-s.

The theorem proves that a half-integer-s spin chain with essentially *any* reasonably local Hamiltonian respecting translational symmetry and rotational symmetry either has zero gap (i.e. "mass") or else has degenerate ground states, corresponding to spontaneously broken parity (i.e. reflection about a site). We will see later that either possibility can be realized for various spin-chain Hamiltonians.

The theorem, though rigorous, is elementary. We begin with a finite chain of even length L and periodic boundary conditions, and a ground state, $\mid \psi_0 >$, respecting parity and rotational invariance. We now construct a state $\mid \psi_1 >$ which is orthogonal to $\mid \psi_0 >$ and has low energy. Explicitly:

$$\langle \psi_1 \left| (H - E_0) \right| \psi_1 \rangle = O\left(\frac{1}{L}\right),$$

where E_0 is the ground state energy. While $\mid \psi_1 >$ is not itself an energy eigenstate, this construction proves that a low-energy eigenstate exists. The state $\mid \psi_1 >$ is constructed by acting with a unitary operator U: $\mid \psi_1 >= U \mid \psi_0 >$. Numbering the sites from $-L/2$ to $L/2-1$, U is defined by:

$$U = \exp\left[\frac{i\pi}{l}\sum_{j=-l}^{l}(j+l)S_j^z\right].$$

Thus we are twisting the j^{th} spin by $\pi(j+l)/l$ about the z axis. This twist varies from 0 to 2π as we travel along the chain, from $-l$ to l. The sites with $|i| > l$ are untwisted. l is chosen to be some number of $O(L)$, for instance, we may choose, $l = L/2-1$. We must prove 2 things. Firstly that $\mid \psi_1 >$ has low energy, and secondly that it is orthogonal to the ground state. The first fact follows from the fact that the relative twist of 2 neighboring sites is $\pi/l << 1$. Since a uniform twist is a symmetry operation and costs no energy at all, a slow twist costs only a small energy. Explicitly,

$$U^\dagger S_i^+ S_{i+1}^- U = \mathrm{e}^{-i(\pi/l)} S_i^+ S_{i+1}^-.$$

Thus, for the Heisenberg Hamiltonian:

$$\begin{aligned}\delta E \equiv & \langle \psi_1 \left| (H - E_0) \right| \psi_1 \rangle \\ = & \frac{J}{2}\sum_i \left\langle \psi_0 \left| \left(\left(\mathrm{e}^{-i(\pi/l)} - 1\right) S_i^+ S_{i+1}^- + \left(\mathrm{e}^{i\pi/l} - 1\right)\right) S_i^- S_{i+1}^+ \right| \psi_0 \right\rangle.\end{aligned}$$

The assumed symmetry of the ground state allows us to rewrite this as:

$$\begin{aligned}\delta E = & J\left(\cos\frac{\pi}{l} - 1\right)\sum_i \langle \psi_0 \left| S_i^+ S_{i+1}^- \right| \psi_0 \rangle \\ = & \frac{2}{3}\mathrm{e}_0\left(\cos\frac{\pi}{l} - 1\right)(2l+2) = O\left(\frac{1}{l}\right).\end{aligned}$$

(e_0 is the ground state energy per link.) We have only given the proof for the Heisenberg Hamiltonian but it generalizes easily to essentially any symmetric local Hamiltonian. δE can be always expressed as the expectation value of some local operator times a quantity of $O(1/l)$.

Nothing in this part of the proof distinguishes integer and half-integer-s. However, we still must proof that $< \psi_1 \mid \psi_0 >= 0$. Otherwise there is danger that $\mid \psi_1 >$ will approach $\mid \psi_0 >$ as $l \to \infty$, and this may not imply anything about the spectrum in the infinite l limit. To prove orthogonality we simply note that $\mid \psi_1 >$ has the opposite parity to $\mid \psi_0 >$. Actually, we use a product of parity and a rotation about the y-axis by π. The combined operation takes:

$$S_i^z \to -S_{-i}^z \text{ and hence } U \to U \exp\left[-2i\pi\sum_{j=-l}^{l} S_j^z\right].$$

But since the sum contains an odd number of spins

$$\begin{aligned}\left[-2i\pi\sum_{j=-l}^{l} S_j^z\right] = & -1(\text{half} - \text{integer} - s) \\ = & +1(\text{integer} - s)\end{aligned}$$

Thus $\mid \psi_1 >$ is parity even or odd for s integer or half-integer, respectively. (Here we are measuring the parity relative to that of the ground state.) In the latter case $\mid \psi_1 >$ is necessarily orthogonal to the ground state.

There are certainly two possibilities left open by this result. At $L \to \infty$, there may be a gapless, particle-like excitation with odd parity. Alternatively, parity may be spontaneously broken with 2 ground states and a gap above each. An example of simple candidate ground states with broken parity are the simple dimer configurations:

(Neighboring pairs of spins are contracted to form singlets:

$$\mid \psi >= \mid \alpha_1\alpha_2\alpha_3\alpha_4\cdots > \varepsilon^{\alpha_1\alpha_2}\varepsilon^{\alpha_3\alpha_4}\ldots)$$

Hamiltonians with have ground states like this are discussed in [4] and references therein. Normally the ground state is unique for a finite length chain, in this situation. The two approximate ground states have some overlap, which goes to zero like $e^{-\text{const}\cdot L}$. This mixing produces a symmetric ground state (essentially the symmetric linear combination of the two) and a low-lying $[\delta E = O(e^{-\text{const}\cdot L})]$ antisymmetric excited state. This is completely analogous to the mixing of the two approximate ground states in a double-well potential in one-dimensional quantum mechanics. The mixing is $\propto e^{-S/\hbar}$, where S is the action of the tunneling configuration.

Although the above arguments constitute a "physicist's proof" that either parity is spontaneously broken or else there are gapless excitations of odd parity, it is non-trivial to make this argument completely rigorous. The needed argument starts with a fixed value of l and lets $L \to \infty$ first. The fact that U is a local operator, although acting on an infinite vector space is essential. If it were only possible to create a low-energy state by spreading it over the whole chain then there is a danger that it might become unobservable, and yet not connected with ground state degeneracy at $L \to \infty$.

Taking the large-s limit we conclude that the $O(3)$ σ-model at $\theta = \pi$ must either have spontaneously broken parity or else vanishing mass. We saw the first possibility occur in the large-n (CP^n) approximation. I will argue later that, in fact, it is the second that actually occurs for $n = 1$. On the other hand, at $\theta = 0$, or integer-s we expect to have a unique ground state, with a finite mass.

6. Spin-1/2 chain by Jordan–Wigner transformation

We have so far been mainly discussing the large-s limit. I now go to the opposite extreme and talk about $s = 1/2$. In this section I use a rather miraculous representation of a spin chain in terms of *spinless* fermions which conceals the SU(2) symmetry. It does not generalize to higher spin or higher dimension. These results will be rederived in a different way in a later lecture. The big advantage of the present method is that it happens to *exact* for the $s = 1/2$ quantum XY model [13], i.e.:

$$H = \sum_i \left[-\frac{1}{2} J \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right) + J_z S_i^z S_{i+1}^z \right], \qquad (5)$$

in the limit $J_z = 0$. We will then do perturbation theory in J_z and try to extrapolate up to $J_z = \pm 1$, the isotropic antiferromagnet and

ferromagnet, respectively. [Note that the sign of the XY term can be reversed by making a rotation by π about the z-axis on the odd sublattice, without affecting the sign of the zz term. We have chosen the sign negative, for convenience in the above.]

The miraculous transformation, discovered by Jordan and Wigner, involves introducing a fermion annihilation operator, ψ_i at each site, and writing:

$$S_i^z = n_i - \frac{1}{2}, \text{ where } n_i = \psi_i^\dagger \psi_i,$$

i.e. we identify spin up or down with the presence or absence of the fermion. We then try to simply represent S_i^- in terms of ψ_i. The formula

$$S_i^- = \psi_i,$$

looks right in that annihilating the fermion takes us from the spin-up to spin-down state. However, S_i^- commutes with the spin operators on the other sites, whereas ψ_i anticommutes with $\psi_j^\dagger$ for $j \neq i$, so this doesn't quite work. We need to add an additional factor to correct the statistics. What works is [13]:

$$S_i^+ = \psi_i \exp\left[i\pi \sum_{j=1}^{i-1} n_j \right].$$

This can be shown to fix up the problem using:

$$\exp\left[i\pi n_j\right] \psi_j = -\psi_j \exp\left[i\pi n_j\right].$$

The nicest aspect of all this is that the nasty string operator $\exp\left[i\pi \sum_{j=1}^{i-1} n_j\right]$, cancels out in the nearest neighbor interaction:

$$H_{XY} \to -\frac{J}{2} \sum_i \left\{ \psi_i^\dagger \exp\left[i\pi n_i\right] \psi_{i+1} + \psi_{i+1}^\dagger \exp\left[i\pi n_i\right] \psi_i \right\}.$$

Since ψ_i creates a state with $n_i = 0$, it follows that:

$$\exp\left(i\pi n_j\right) \psi_i = \psi_i,$$

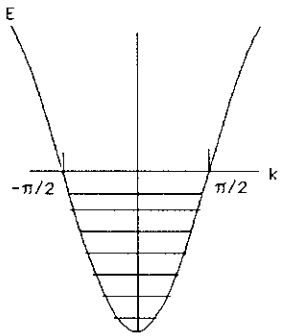
and so:

$$H_{XY} = -\frac{J}{2} \sum_i \left[\psi_{i+1}^\dagger \psi_i + \text{h.c.} \right].$$

This is simply a free spinless fermion Hamiltonian! We may immediately find the ground state by Fourier transforming:

$$H \rightarrow -J \sum_k \cos(ak) \psi_k^\dagger \psi_k .$$

The ground state has a half-filled Fermi sea, corresponding to total magnetization: $M \equiv \Sigma_i S_i^z = 0$. The Fermi surface is at $\pm\pi/2a$. Although various exact results can be derived fairly easily, we now pass to the continuum limit, suitable for describing the low-energy properties of the system. For more general models, it is *only* the continuum limit which will be accessible. The $s = 1/2$ XY chain provides a valuable example where the validity of the continuum limit results can be checked against an exact lattice solution.



The low-energy excitations involve creating holes just below the Fermi surface and electrons just above it. Thus we are only concerned with the ψ_k operators for $k \approx \pm\pi/2a$ (where "a" is the lattice spacing). Essentially we may truncate the Fourier expansion and only keep k in the range

$$\left| k \pm \frac{\pi}{2a} \right| \leq \Lambda,$$

where Λ is an ultraviolet cut-off which can be taken to be $<< 2\pi/a$, where a is the lattice spacing. This will be sufficient to study the physics at length scales $>> 1/\Lambda$. Thus we write:

$$\frac{\psi(x)}{\sqrt{a}} \approx \mathrm{e}^{i(\pi/2a)x} \psi_{\mathrm{L}}(x) + \mathrm{e}^{-i(\pi/2a)x} \psi_{\mathrm{R}}(x),$$

where ψ_{R} and ψ_{L} are slowly varying on the lattice scale and contain the Fourier modes near $\pm\pi/2a$ respectively. It is convenient to define

continuum Fourier modes:

$$a(k) = \psi\left(\frac{\pi}{2a} + k\right), \quad b^\dagger(k) = \psi\left(\frac{\pi}{2a} - k\right), \quad \text{for } k > 0,$$

where a and b are electron and hole (or positron, in the relativistic limit) annihilation operators. Thus ψ_{R} annihilate right-moving electrons and creates right-moving holes:

$$\psi_{\mathrm{R}} = \frac{1}{\sqrt{a}} \sum_{k>0} \left[\mathrm{e}^{-ikx} a(k) + \mathrm{e}^{ikx} b^\dagger(k) \right].$$

Likewise, ψ_{L} is written in terms of the left-moving electrons and holes:

$$\psi_{\mathrm{L}} = \frac{1}{\sqrt{a}} \sum_{k<0} \left[\mathrm{e}^{-ikx} a(k) + \mathrm{e}^{ikx} b^\dagger(k) \right].$$

In the continuum approximation we linearize the dispersion relation near the Fermi surface. This leads to the continuum Hamiltonian

$$H = iv \int \mathrm{d}x \left[\psi_{\mathrm{R}}^\dagger \frac{\mathrm{d}}{\mathrm{d}x} \psi_{\mathrm{R}} - \psi_{\mathrm{L}}^\dagger \frac{\mathrm{d}}{\mathrm{d}x} \psi_{\mathrm{L}} \right],$$

where $v = Ja$, the Fermi velocity.

Thus we obtain a Lorentz-invariant massless Dirac Fermion field theory in the low-energy approximation, with "velocity of light" v. In what follows, we generally follow the particle physics convention of setting $v = 1$; it can always be restored by dimensional analysis.

All long wave-length, low energy properties of the theory can now be read off from the Lorentz invariant field theory. Consider for example the correlation function

$$G^z(x,t) \equiv \langle S^z(x,t) S^z(0,0) \rangle .$$

We first write:

$$S_j^z = \psi^\dagger(x)\psi(x) - \frac{1}{2}.$$

Note that $< 0 \mid \psi^\dagger \psi \mid 0 >= 1/2$, so, we may write:

$$S_j^z =: \psi^\dagger(x)\psi(x) :,$$

where we write the electron and hole annihilation operators on the right. In the continuum limit this becomes:

$$\frac{S_j^z}{a} \approx \left(:\psi_{\rm L}^\dagger\psi_{\rm L}: + :\psi_{\rm R}^\dagger\psi_{\rm R}:\right) + (-1)^j\left(\psi_{\rm R}^\dagger\psi_{\rm L} + \psi_{\rm L}^\dagger\psi_{\rm R}\right). \qquad (6)$$

Thus we see that the long-range correlation function has two parts: one uniform and one alternating:

$$\begin{aligned} G^z(x) \approx &[\left\langle :\psi_{\rm L}^\dagger(x)\psi_{\rm L}(x)::\psi_{\rm L}^\dagger(0)\psi_{\rm L}(0):\right\rangle \\ &+\left\langle :\psi_{\rm R}^\dagger(x)\psi_{\rm R}(x)::\psi_{\rm R}^\dagger(0)\psi_{\rm R}(0):\right\rangle] \\ &+(-1)^{x/a}[\left\langle\psi_{\rm L}^\dagger(x)\psi_{\rm R}(x)\psi_{\rm R}^\dagger(0)\psi_{\rm L}(0)\right\rangle \\ &+\left\langle\psi_{\rm R}^\dagger(x)\psi_{\rm L}(x)\psi_{\rm L}^\dagger(0)\psi_{\rm R}(0)\right\rangle]. \end{aligned}$$

This feature of correlation functions will be generic for all our calculations. It reflects the antiferromagnetic nature of the problem; there are low-energy modes at momenta near zero and near π/a. It means, in general, that we have twice as much work to do: twice as many critical exponents to calculate, etc. In this case, we obtain:

$$G^z(x,t) = -\left(\frac{1}{4\pi}\right)^2\left[\left(\frac{1}{x_-^2}+\frac{1}{x_+^2}\right)+(-1)^{x/a}\frac{2}{x_-x_+}\right],$$

i.e. both uniform and alternating pieces decay as $1/x^2$.

The transverse correlation function:

$$G^{-+}(x,t) \equiv \left\langle S^+(x,t)S^-(0,0)\right\rangle,$$

appears more complicated because of the string operator. Remarkably, it becomes a local operator in the continuum limit, upon bosonizing. A brief review of the relevant formulae, primarily for the purpose of establishing my notation, is given in the Appendix. More details are given in Ginsparg's lectures in this series.

$$\sum_{j=1}^{i-1} n_j \to \int_0^x {\rm d}x'\left(J_0+\frac{1}{2a}\right) = -\frac{1}{\sqrt{\pi}}[\varphi(x)-\varphi(0)]+\frac{x}{2a}.$$

We write the string operator in a manifestly hermitian way as:

$$\begin{aligned} \exp\left[i\pi\sum_{j=1}^{i-1}n_j\right] \propto &{\rm e}^{i\pi x/2a}\exp\left[-i\sqrt{\pi}\left(\varphi_{\rm L}(x)+\varphi_{\rm R}(x)\right)\right] \\ &+{\rm e}^{-i\pi x/2a}\exp\left[i\sqrt{\pi}\left(\varphi_{\rm L}(x)+\varphi_{\rm R}(x)\right)\right]. \end{aligned}$$

Also bosonizing the single fermion operators we obtain:

$$\begin{aligned} S^-(x) \propto &\exp\left[i\sqrt{\pi}\left(\varphi_{\rm R}-\varphi_{\rm L}\right)\right] + \text{constant}\ {\rm e}^{i\pi x/a} \\ &\times\left\{\exp\left[-i\sqrt{\pi}\left(3\varphi_{\rm L}+\varphi_{\rm R}\right)\right]+\exp\left[-i\sqrt{\pi}\left(3\varphi_{\rm R}+\varphi_{\rm L}\right)\right]\right\} \quad (7) \end{aligned}$$

Thus:

$$G^{+-}(\mathbf{x}) \propto (-x_-x_+)^{-1/4}\left[1+\text{constant}(-1)^{x/a}\left(x_+^{-2}+x_-^{-2}\right)\right].$$

These results all agree with direct calculations not using bosonization [13,15].

The real power of the field theory methods is that they can be extended away from the $J_z = 0$ case [16]. The Jordan-Wigner transformation turns the general $U(1)$ invariant model of eq. (5) into a spinless lattice fermion model with a nearest neighbor interaction:

$$H = \sum_i\left[-\frac{J}{2}\left(\psi_i^\dagger\psi_{i+1}+\text{h.c.}\right)+J_z\left((\psi_i^\dagger\psi_i-\frac{1}{2}\right)\left(\psi_{i+1}^\dagger\psi_{i+1}-\frac{1}{2}\right)\right].$$

To study the low energy behavior of this model, we pass to the continuum limit. Using eq. (5) the interaction term gives:

$$\begin{aligned} H_{\rm int} \approx &J_z a\int {\rm d}x\left[\left(J_{\rm L}+J_{\rm R}\right)^2-\left(\psi_{\rm L}^\dagger\psi_{\rm R}+\psi_{\rm R}^\dagger\psi_{\rm L}\right)^2\right] \\ H_{\rm int} = &J_z a\left\{\left(J_{\rm L}^2+J_{\rm R}^2\right)+4J_{\rm L}J_{\rm R}-2\left[\left(\psi_{\rm L}^\dagger\psi_{\rm R}\right)^2+\text{h.c.}\right]\right\} \end{aligned}$$

The three different terms in $H_{\rm int}$ have very different effects. The first term is non-Lorentz-invariant. It actually comes from a term: $J_{\rm L}(x)J_{\rm L}(x+a)+J_{\rm R}(x)J_{\rm R}(x+a)$. Up to higher-derivative terms this just gives the free fermion Hamiltonian:

$$\begin{aligned} J_{\rm L}(x)J_{\rm L}(x+a) \approx &:J_{\rm L}(x)J_{\rm L}(x):+\frac{i}{2\pi a} \\ &\left[\psi_{\rm L}^\dagger(x+a)\psi_{\rm L}(x)-\psi_{\rm L}^\dagger(x)\psi_{\rm L}(x+a)\right]+\text{constant} \\ \approx &-\frac{i}{\pi}\psi_{\rm L}^\dagger\frac{\rm d}{{\rm d}x}\psi_{\rm L}+\text{constant} \\ J_{\rm R}J_{\rm R} \approx &\frac{i}{\pi}\psi_{\rm R}^\dagger\frac{\rm d}{{\rm d}x}\psi_{\rm R}+\text{constant}. \end{aligned}$$

Thus the velocity of "light" is renormalized:

$$v \to a\left(J + J_z/\pi\right).$$

The third interaction term above is naively zero by Fermi statistics. Thus I will drop it now, but will have more to say about it later. The second term gives a Lorentz-invariant interaction. The corresponding Lagrangian, (upon setting $v = 1$ once again) is:

$$L = -i\left(\psi_{\mathrm{L}}^{\dagger}\partial_{-}\psi_{\mathrm{L}} + \psi_{\mathrm{R}}^{\dagger}\partial_{+}\psi_{\mathrm{R}}\right) - 4\frac{J_z}{J}J_{\mathrm{L}}J_{\mathrm{R}}.$$

This is the Thirring model and it can be solved by bosonization. The interaction term is just proportional to the free boson Lagrangian:

$$J_{\mathrm{L}}J_{\mathrm{R}} = -\frac{1}{4\pi}\partial_{-}\varphi\partial_{+}\varphi = \frac{1}{4\pi}\partial_{\mu}\varphi\partial^{\mu}\varphi.$$

Thus

$$L \approx \frac{1}{2}\left(1 + 2J_z/\pi J\right)\partial_{\mu}\varphi\partial^{\mu}\varphi.$$

Apparently, the only effect of the interaction is to rescale φ and the continuum, critical theory is simply a free boson, for all J_z! We will come back and examine this point a little more critically later, but for the moment let's see what it implies about the correlation functions. We may use the bosonized form for the spin operators in the continuum limit but we must rescale the boson. It is convenient to define a rescaling factor:

$$R \equiv \left[\left(1 + 2J_z/\pi J\right)/4\pi\right]^{1/2} \tag{8}$$

so that

$$\sqrt{4\pi}\varphi \to \varphi/R$$

$$S^z(x) \approx \frac{1}{2\pi R}\frac{\partial\varphi}{\partial x} + (-1)^{x/a}\text{constant}\cdot \mathrm{e}^{i\varphi/R}$$

$$G^z(x,t) \approx -\frac{1}{16\pi^3R^2}\left(\frac{1}{x_{-}^2} + \frac{1}{x_{+}^2}\right) + \text{constant}\cdot\left(x^2 - t^2\right)^{-1/4\pi R^2}.$$

To obtain G^{+-} we must understand how φ_{L} and φ_{R} are effected by the rescaling of φ. Because these fields are non-commuting and actually contain linear combinations of φ with its conjugate momentum, Π, the answer is non-trivial. To preserve the canonical commutation relations Π is rescaled in the opposite way:

$$\varphi \to \varphi/\sqrt{4\pi}R, \quad \Pi \to \sqrt{4\pi}R\varphi.$$

The fields φ_{L} and φ_{R} can be written:

$$\varphi_{\mathrm{R}}(x) = \frac{1}{2}\left[\varphi(x) + \int_{-\infty}^{x}\mathrm{d}x'\Pi(x')\right]$$

$$\varphi_{\mathrm{L}}(x) = \frac{1}{2}\left[\varphi(x) - \int_{-\infty}^{x}\mathrm{d}x'\Pi(x')\right].$$

Thus the fields φ_{L} and φ_{R} mix under the canonical transformation:

$$\left(\varphi_{\mathrm{L}}, \varphi_{\mathrm{R}}\right) \to \left(\varphi_{\mathrm{L}}, \varphi_{\mathrm{R}}\right)\begin{bmatrix}\cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta\end{bmatrix}$$

where $\mathrm{e}^{\beta} \equiv 1/\sqrt{4\pi}R$. Making this rescaling in eq. (7) gives:

$$S^{-}(x) \propto \exp[-i2\pi R\tilde{\varphi}] + \text{constant}\ (-1)^{x/a} \times\left\{\exp i\left[\frac{\varphi}{R} - 2\pi R\tilde{\varphi}\right] + \exp i\left[-\frac{\varphi}{R} - 2\pi R\tilde{\varphi}\right]\right\} \tag{9}$$

where I have defined $\tilde{\varphi}$ by:

$$\tilde{\varphi} \equiv \varphi_{\mathrm{L}} - \varphi_{\mathrm{R}}.$$

From this we can read off the correlation exponents:

$$G^{+-}(\mathbf{x}) \propto (-x_{+}x_{-})^{-\pi R^2} + (-1)^{x/a}\text{constant}\ (-x_{-}x_{+})^{-(1/R - 2\pi R)^2/4\pi}\left(\frac{1}{x_{-}^2} + \frac{1}{x_{+}^2}\right).$$

Altogether we have predicted 4 correlation exponents for the uniform and alternating parts of G^z and G^{+-} (8 if we count the x_{+} and x_{-} parts separately) plus one critical amplitude, that of the uniform part of G^z. Let's now pause and ask if we should really believe this calculation. In any such problem it is always important to ask if we have left anything out; are there are other terms that can somehow sneak into the Lagrangian which could change the conclusions? In general, when we

start with a lattice problem we generate infinite numbers of interactions. If we are lucky, most of them are irrelevant and can be ignored. But we should always consider carefully what the relevant operators are. In general, any relevant operator that is not forbidden by symmetry will be generated by one effect or another.

If we accept that the critical theory can be written in terms of the boson φ then let us ask what relevant operators can be added to L. The only Lorentz-invariant relevant interactions are constructed from $\cos\beta\varphi$ and $\cos\beta\tilde{\varphi}$ which have dimension $\beta^2/4\pi$ and thus are relevant for $\beta \leq \sqrt{8\pi}$. What symmetries, if any, prevent them from appearing? One important symmetry in the problem is the $U(1)$ invariance of the spin chain. How does this act on the bosons? There are various ways of seeing that it corresponds to

$$\tilde{\varphi} \to \tilde{\varphi} + \text{constant}, \quad \varphi \to \varphi.$$

For instance note that all terms in the continuum representation of eq. (7) for S^- contain $(-i2\pi R\tilde{\varphi})$ in the exponent whereas the formula for S^z is written in terms of φ only. In the fermion language, we see that to multiply both ψ_L and ψ_R by the same phase, we must shift φ_L and φ_R by opposite constants. Thus the $U(1)$ symmetry forbids any interaction of the form $\cos\beta\tilde{\varphi}$, and we only need to worry about $\cos\beta\varphi$. The first restriction on φ -interactions comes from the fact that φ must be thought of as a periodic field [17]:

$$\varphi \leftrightarrow \varphi + 2\pi R.$$

This can be seen from the fact that the spin operators, the only observables in the theory, are expressed in terms of $e^{i\varphi/R}$. Thus operators of the form $\cos\beta\varphi$ are single-valued only for $\beta = n/R$, for integer n. Only single-valued operators should appear in the Lagrangian. This however, is not enough to forbid relevant interactions; at the $J_z = 0$ point $\cos\sqrt{4\pi}\varphi$ would be permitted. The final needed symmetry is translation by one site in the spin chain. From the continuum limit definition of ψ_L and ψ_R we see that this corresponds to a discrete *internal* symmetry in the continuum limit which is distinct from the translational symmetry:

$$\psi_L \to i\psi_L, \quad \psi_R \to -i\psi_R.$$

This corresponds to the symmetry:

$$\varphi_L \to \varphi_L + \pi R/2, \quad \varphi_R \to \varphi_R + \pi R/2, \quad \text{or}$$
$$\varphi \to \varphi + \pi R.$$

This further restricts the allowed interactions to $\cos 2n\varphi/R$. Thus we should expect the theory to be in a massless phase for $R < 1/\sqrt{2\pi}$. This second, Z_2 symmetry is actually a sub-group of the chiral $U(1)$ symmetry which rotates ψ_L and ψ_R by independent phases, and thus shifts φ and $\tilde{\varphi}$ by independent constants. This is a symmetry of the free boson critical theory. We will argue later that any left-right symmetric critical point with a continuous symmetry necessarily has a chiral symmetry. In general, some sub-group of this chiral symmetry is enough to forbid all relevant operators.

We obtained the formula:

$$R \equiv \left[\frac{1}{4\pi}\left(1 + \frac{2J_z}{\pi J}\right)\right]^{1/2},$$

from the bosonized interacting theory. This suggests that the massless phase should persist for increasing J_z up to the point

$$\frac{J_z}{J} = \frac{\pi}{2},$$

which is beyond the isotropic point. However, this formula for R is not reliable except for very small J_z/J.

$$R = \frac{1}{\sqrt{4\pi}}\left[\frac{J_z}{\pi J} + O\left(J_z^2\right)\right].$$

The various non-renormalizable interactions can lead to renormalization of R in a way which is not predicted by the continuum field theory. However, by the above symmetry analysis, this is *all* that these interactions can do. R should increase in some way with J_z and eventually reach the value $1/\sqrt{2\pi}$, at which point the $\cos\sqrt{8\pi}\varphi$ interaction appears and produces a mass. The point where this occurs is in fact the isotropic one, $J_z = J$. One way of seeing this is to observe that the $R = 1/\sqrt{2\pi}$ theory has SU(2) symmetry [17]. Note that G^z and G^{+-} become the same at this value of R (after taking into account the change of sign of S^x, S^y on the odd sub-lattice, mentioned near the beginning of this chapter).

We might ask: where does this $\cos 2\varphi/R$ interaction come from? It can actually be traced back to the third term in H_{int}: $-2J_z a[(\psi_L^\dagger\psi_R)^2 + h.c.]$, which we derived earlier. This is actually the operator:

$$\psi_L^\dagger(x)\psi_R(x)\psi_L^\dagger(x+a)\psi_R(x+a).$$

Naively, this seems to vanish in the limit a $a \to 0$ leaving only irrelevant derivative terms. However, if we use the bosonized form, we obtain:

$$e^{i\varphi(x)/R} \cdot e^{i\varphi(x+a)/R} \propto e^{2i\varphi(x)/R} + \text{derivative terms.}$$

At weak coupling where $R \approx 1/\sqrt{4\pi}$, the $e^{2i\varphi(x)/R}$ operator is irrelevant and can be dropped. Indeed, in this limit it could be identified with a dimension four operator containing four powers of the Fermion fields and two derivatives. However, as R increases it eventually becomes relevant.

Apart from a mass appearing in the theory for $R > 1/\sqrt{2\pi}$, we should also expect one to appear upon addition of even an infinitesimal interaction which breaks either of the two important symmetries mentioned above, namely the $U(1)$ rotational symmetry and translation by one site. A $U(1)$ breaking interaction of the form:

$$H = \sum_i \left[\frac{J}{2} \left(S_i^+ S_{i+1}^- + \text{h.c.} \right) + J_z S_i^z S_{i+1}^z + J_{XY} \left(S_i^- S_{i+1}^- + \text{h.c.} \right) \right],$$

is of special interest. The new interaction is Jordan-Wigner transformed into:

$$H' = J_{XY} \sum_i \left(\psi_i \psi_{i+1} + \text{h.c.} \right).$$

In the continuum limit this becomes:

$$H' = 2iJ_{XY} \sum_i \left(\psi_{\mathrm{L}} \psi_{\mathrm{R}} - \psi_{\mathrm{R}}^\dagger \psi_{\mathrm{L}}^\dagger \right).$$

The bosonized form is $\sin \sqrt{4\pi}\tilde{\varphi}$, or $\sin 4\pi R\tilde{\varphi}$, in general. This is relevant along the whole critical line from $J_z = -1$ to $+1$. Its dimension is $x = 4\pi R^2$.

In general when a relevant operator is present we expect to induce a mass. The scaling of the mass with the relevant coupling constant is determined from the mass-dimensions of the coupling constant, i.e.

$$m \propto J_{XY}^{1/(2-x)}.$$

This formula actually allows an exact determination of R as a function of J_z from the Bethe ansatz.

This is the first time that I have mentioned the Bethe ansatz so far. Although the Bethe ansatz gave some exact results many years ago, it is

difficult to get a complete picture of the critical behavior of a spin chain this way; in particular, the correlation functions cannot be calculated. However, it does provide many useful checks on the field theory arguments, and that is the spirit in which I use it. The spin chain is solvable by the Bethe ansatz for arbitrary J_z and J_{XY}. The exponent $1/(2-x)$ giving the scaling of the mass with J_{XY} can be calculated exactly for arbitrary J_z along the critical line [18]. This determines $R(J_z)$ to be:

$$R = \left\{ \frac{1}{2\pi} - \frac{1}{2\pi^2} \cos^{-1} \left(\frac{J_z}{J} \right) \right\}^{1/2}.$$

Thus $R = 1/\sqrt{4\pi}(1 + J_z/\pi J)$ at small J_z as obtained from perturbation theory in J_z/J in the continuum theory. Furthermore, $J_z \to 1/\sqrt{2\pi}$ at the isotropic point, $J_z = J$, as expected from the SU(2) symmetry at this value of R.

Another way of inducing a mass is to break the symmetry of translation by one site; i.e. to add alternating interactions.

$$H \to H + J_A \sum_i (-1)^i \left(S_i^+ S_{i+1}^- + \text{h.c.} \right).$$

This gives the lattice fermion interaction:

$$H' = J_A \sum_i (-1)^i \left((\psi_i^\dagger \psi_{i+1} + \text{h.c.} \right).$$

In the continuum limit this picks out the alternating piece of the spin-spin operator:

$$H' = iJ_A \left((\psi_{\mathrm{L}}^\dagger \psi_{\mathrm{R}} - \text{h.c.} \right).$$

This gives the interaction $\cos \varphi/R$, which is relevant along the whole critical line.

It is also interesting to ask what happens for $J_z < 0$. R starts to decrease and eventually reaches 0 at $J_z = -1$, the isotropic ferromagnetic point. This represents a type of singular point where the $(\partial_\mu \varphi)^2$ term disappears from the action. We can get a clue to what this means by looking at the behavior of the spin-correlation functions at $R \to 0$. We see that the alternating part of the correlation functions go to zero rapidly as $(-x_+ x_-)^{-1/4\pi R^2}$ and the uniform part blows up, corresponding to ferromagnetic order. This ordered phase does not have a Lorentz-invariant field theory description. It is possible in one dimension because the order parameter, the total magnetization, commutes with the

Hamiltonian. Indeed the ferromagnetic ground state is trivial, even for the *quantum* spin chain.

Having performed this symmetry analysis of the critical theory we can now predict, with some confidence, the behavior of essentially *any* Hamiltonian of the type we are considering. Massless behavior should only occur (barring some fine-tuning of parameters) when the theory has both $U(1)$ symmetry and the additional Z_2 symmetry of translation by one site. Even then, it only occurs for some finite range of parameters, corresponding to $0 < R < 1/\sqrt{2\pi}$.

7. The Hubbard model-large-U limit and spin-density-wave mean field theory

I now want to change topics and discuss a model for itinerant electrons which is more general than the Heisenberg model. Apart from its intrinsic interest, one of the motivations will be obtain a different approach to bosonizing the antiferromagnetic chain which makes the SU(2) symmetry more manifest. The model is similar to the one we obtained from applying the Jordan-Wigner transformation to the spin chain except that now the fermions are supposed to be real electrons that carry a spin index, α. I will begin with some general remarks about the model in arbitrary dimension and discuss the standard mean field theory approach, before reverting to $(1+1)$ dimensions.

$$H_H = \sum_{<ij>} \left(t\psi_i^{+\alpha}\psi_{\alpha,j} + \text{h.c.}\right) + U\sum_i n_{i\uparrow}n_{i\downarrow}.$$

Here $n_{i\downarrow}$ and $n_{i\uparrow}$ are the electron numbers $\psi_i^{+\alpha}\psi_{\alpha i}$ where $\alpha = 1$ or 2 (not summed). This is a standard model for electrons in well-localized atomic orbitals with a probability t for transitions between neighboring atoms. The constant U, which is normally positive, represents a highly screened Coulomb repulsion between electrons. It costs energy U to put 2 electrons on the same site. This very simplified form assumes that the screening length is on the order of the lattice spacing. In the extended Hubbard model a nearest neighbor repulsion is also maintained. The on-site interaction can be written in a number of equivalent ways, including:

$$H' = \frac{U}{2}\sum_i n_i^2 \quad \text{or,} \quad -\frac{2U}{2}\sum_i \mathbf{S}_i^2.$$

Here n_i is the total electron number:

$$n_i = \psi_i^{+\alpha}\psi_{\alpha i} \text{ (summed over } \alpha),$$

and $\mathbf{S}_i$ is the electron spin operator:

$$\mathbf{S}_i = \psi_i^{+\alpha}\tfrac{1}{2}\sigma_a^\beta\psi_{\beta i}.$$

The various ways of writing H' differ by terms proportional to $\sum_i n_i$, the total electron number. This is a conserved quantity so such a term just shifts the chemical potential.

The strong-coupling limit of the Hubbard Model at half-filling (the total number of electrons equals the number of sites) is of special interest. Setting $t = 0$, we see that any of the 2^V states (V is the volume, or the number of sites) with one particle per site is a ground state. Of course, this large degeneracy is broken by the hopping term. This problem may be studied in degenerate perturbation theory. A single application of the hopping term always produces a doubly occupied site and so increases the energy by U. Note that the system is an electric insulator in the strong-U limit. Transport of an electron by L sites only occurs in L^{th} order perturbation theory and is suppressed by a factor of $(t/U)^L$. Wave-functions are exponentially localized. This insulating behavior at half-filling could not occur without interactions. Trivial insulators always have filled bands. This type of non-trivial insulator, known as a Mott–Hubbard insulator is often observed. It occurs, for example, in compounds closely related to high-T_c superconductors. Mixing of the ground states occurs in second-order degenerate perturbation theory in t. Let us consider the mixing induced by the hopping term on a particular link:

$$\left\langle \downarrow\uparrow \left| H_{\text{int}}\left(E_0 - H_0\right)^{-1} H_{\text{int}} \right| \downarrow\uparrow \right\rangle = -\frac{2t^2}{U},$$

$$\left\langle \downarrow\uparrow \left| H_{\text{int}}\left(E_0 - H_0\right)^{-1} H_{\text{int}} \right| \uparrow\downarrow \right\rangle = \frac{2t^2}{U}.$$

(The factor of 2 arises because there are two possible intermediate states with both electrons on either of the 2 sites. The minus sign occurs from anti-commuting the creation operators.) The matrix element is zero for parallel spins, since the 2 electrons cannot go onto the same site in the intermediate state. We may write down an effective Hamiltonian in the space of singly occupied states which has the same matrix elements [19].

$$H_{\text{eff}} = J\sum_{<ij>}\left[\mathbf{S}_i\cdot\mathbf{S}_j - \frac{1}{4}\right],$$

where,

$$J = \frac{4t^2}{U}.$$

Thus Mott–Hubbard insulators are expected to be antiferromagnets. This is also frequently observed; in fact this is probably the main mechanism responsible for antiferromagnetism in metals. The magnetic interaction results from an exchange process in which neighboring electrons share each others orbitals. This occurs more efficiently for anti-parallel spins due to the Coulomb repulsion and Fermi statistics.

One interesting question is whether or not the insulating antiferromagnetic behavior sets in at arbitrarily small U or whether there is one or more phase transitions at finite U. An insulating, antiferromagnetic phase at small U/t is called a spin-density wave. One imagines almost free electrons interacting with a self-consistently produced Néel spin-density wave. We start with the Hubbard interaction written in magnetic form:

$$H_{\text{int}} = -\frac{2}{3}U \sum_i \left[\frac{1}{2}\psi_i^{+\alpha}\boldsymbol{\sigma}_\alpha^\beta \psi_{i\beta}\right]^2.$$

We then rewrite the quartic fermion interaction in terms of a triplet of bosons, $\mathbf{S}_i$:

$$H_{\text{int}} = \frac{4U}{3}\sum_i \left[\frac{1}{2}\mathbf{S}_i^2 - \mathbf{S}_i.\left(\frac{1}{2}\right)\psi_i^{+\alpha}\boldsymbol{\sigma}_\alpha^\beta\psi_{i\beta}\right].$$

$\mathbf{S}_i$ is a quantum variable but its equation of motion is simply:

$$\mathbf{S}_i = \frac{1}{2}\psi_i^{+\alpha}\boldsymbol{\sigma}_\alpha^\beta\psi_{i\beta}.$$

The other term in the Hamiltonian, quadratic in ψ, is:

$$H_0 - \mu N = \sum_{\mathbf{k}} \psi^\dagger(\mathbf{k})\psi(\mathbf{k})\varepsilon(\mathbf{k}),$$

where $\varepsilon(\mathbf{k})$ gives the free electron dispersion relation. In principle, we now integrate out the electrons to obtain an effective action for the spin variables, $\mathbf{S}_i$. The mean field theory consists of searching for a classical saddle-point for the $\mathbf{S}_i$. A spin-density wave solution corresponds to

$\mathbf{S}_i = \pm\mathbf{m}$, where $\mathbf{m}$ is the sub-lattice magnetization density and we take the + or − sign on one or the other of the two sub-lattice, the effective quadratic fermion Hamiltonian becomes:

$$H_\pm = \sum_{\mathbf{k}} \left\{ \left[\psi_e^\dagger(\mathbf{k})\psi_o(\mathbf{k}) + \text{h.c.}\right]\varepsilon(\mathbf{k}) \pm \frac{2Um}{3}\left[\psi_e^\dagger(\mathbf{k})\psi_e(\mathbf{k}) - \psi_o^\dagger(\mathbf{k})\psi_o(\mathbf{k})\right]\right\}.$$

We may diagonalize the 2×2 matrix to find:

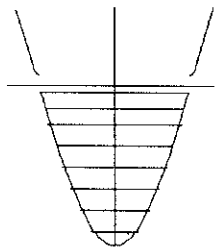
$$E(\mathbf{k}) = \pm\left[\varepsilon(\mathbf{k})^2 + \left(\frac{2Um}{3}\right)^2\right]^{1/2}.$$

Notice that the antiferromagnetic order produces a gap in the electron dispersion relation. This is the mean-field version of the Mott–Hubbard insulating behavior. We expect additional gapless Goldstone modes in the spin-density wave state, characterized by fluctuations of the spin-density $\mathbf{S}_i$.

We may now find m, in our mean field approximation by minimizing the total ground state energy:

$$E_0 = V\left(\frac{2U}{3}\right)m^2 + 2\sum_{\mathbf{k}:\ E<0} E(\mathbf{k}).$$

(The factor of 2 arises from the 2 spin components.) The opening of a gap lowers the energy of the filled Fermi sea, favoring a non-zero m.



However, for $U << t$, the first term in E_0 makes m become very small. Differentiating E_0 with respect to m^2, gives the equation:

$$\frac{3V}{2U} = \sum_{\mathbf{k}} \frac{1}{|E(\mathbf{k})|}.$$

Whether or not the spin-density wave sets in at arbitrarily small U (in the mean field approximation) depends on whether or not the right-hand side is infinite at $m = 0$. Let us first consider the one-dimensional case. The equation becomes:

$$\frac{3}{2U} = \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}k}{2\pi} \left[(2t \cos k)^2 + \left(\frac{2Um}{3} \right)^2 \right]^{-1/2}$$

This equation has a simple solution in the weak-coupling limit. There the integral is dominated by the region near the zeroes of $E(\mathbf{k})$, i.e. the edge of the reduced Brillouin zone $k = \pm\pi/2$.

$$\frac{3}{2U} \approx \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \left[(2tk)^2 + \left(\frac{2Um}{3} \right)^2 \right]^{-1/2} \approx \frac{1}{4\pi t} \ln \frac{3t}{Um}.$$

Thus we obtain a non-zero m for any finite U with

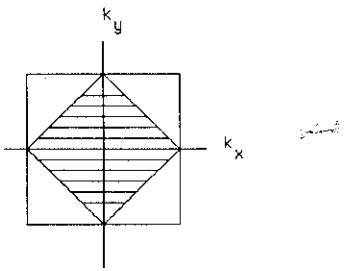
$$m \propto \frac{t}{U} \mathrm{e}^{-6\pi t/U},$$

in the weak-coupling limit. This result, in $D = 1$, is quite insensitive to the form of the kinetic energy, $\varepsilon(k)$. The fermi surface at $k = \pm\pi/2$ always produces a logarithmic divergence.

The situation in higher dimensions is a bit more complicated. If we consider a nearest neighbor hopping term on a $D = 2$ square lattice, we obtain a similar result. The fermi surface for half-filling, defined by the zeroes of

$$\varepsilon(\mathbf{k}) = 2t \left(\cos kx + \cos k_y \right)$$

is a square rotated by $\pi/4$ relative to the k_x axis:



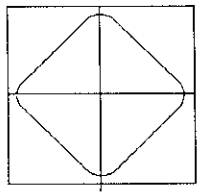
Breaking the integral $\mathrm{d}^2\mathbf{k}$ up into $\mathrm{d}k_{\|}$ parallel to the Fermi surface and $dk_{\perp}$ perpendicular to it, the integral becomes, at $m = 0$

$$L_f \int \mathrm{d}k_{\perp}/k_{\perp}$$

where L_f is the length of the Fermi surface. Thus we again get a logarithmic divergence. Actually, there is a $(\ln m)^2$ divergence at the corners of the Brillouin zone, giving:

$$m \propto \frac{t}{U} \exp \left[-\pi (\frac{3t}{U})^{1/2} \right].$$

Again, the mean field theory predicts spin-density wave order for arbitrarily weak coupling. However, a second nearest neighbor coupling distorts the Fermi surface away from its perfect square shape, and removes the infra-red divergence. (The integrand only vanishes at discrete points.)



Thus a finite-U transition will occur. A similar result holds in $D = 3$. Changing the filling factor from $1/2$ also removes the infra-red singularity (even in $D = 1$) and produces a finite U transition. At negative U the natural mean field theory gives either a superconducting.

$$< \psi_{\downarrow} \psi_{\uparrow} > \neq 0$$

or charge-density wave:

$$\langle n_i \rangle = \text{constant} \pm \text{constant},$$

state.

Of course this is only a mean field theory. Most of these conclusions must, in fact, be wrong in $D = 1$, if not higher D. Antiferromagnetic order cannot occur in $D = 1$ due to Coleman's theorem. However, a gap for charge excitations is always possible, and indeed must occur at large enough U. At large U the low-energy theory should just be the $s = 1/2$ quantum spin chain, discussed above. The question of whether or not a charge-gap sets in at arbitrarily small U can be addressed using similar bosonization techniques to those used earlier. Apart from telling us about what happens to the charge excitations this approach will give us a manifestly SU(2) symmetric bosonized form of the quantum spin chain.

8. Non-abelian bosonization of the Hubbard model

We take the continuum limit precisely as for the spinless fermion model, defining left and right moving fermions for each spin, α. Thus the hopping term gives a Lorentz-invariant free Dirac fermion theory with 2 "flavors", the spin components. (Note that the electron spin appears as an internal quantum number in the $(1+1)$ dimensional field theory; there is no intrinsic spin in one dimension.) The velocity of "light" in this case is $v = 2a|t|$. The continuum free fermion theory (i.e. at $U = 0$) has chiral $U(1)$ and SU(2) symmetries. Operating on the left fermions we have a $U(1)$ charge symmetry:

$$\psi_{\mathrm{L}i\alpha} \to \mathrm{e}^{i\theta}\psi_{\mathrm{L}i\alpha},$$

and an SU(2) spin symmetry:

$$\psi_{\mathrm{L}i\alpha} \to g_\alpha^\beta \psi_{\mathrm{L}i\beta}.$$

Here, g_β^α is an SU(2) matrix. Likewise we may make *independent* transformations on the ψ_{R}'s. The charge and spin of the left and right-moving fermions are separately conserved. Corresponding to these symmetries we have conserved currents. The light-cone components of the currents can be written:

$$J_{\mathrm{L,R}} \equiv :\psi_{\mathrm{L,R}}^{\dagger i\alpha}\psi_{\mathrm{L,R}i\alpha}:,$$
$$\mathbf{J}_{\mathrm{L,R}} \equiv \psi_{\mathrm{L,R}}^{\dagger i\alpha}(1/2)\boldsymbol{\sigma}_\alpha^\beta\psi_{\mathrm{L,R}i\beta},$$

The energy-momentum tensor can be written in a form quadratic in currents:

$$T_{\mathrm{L}} = \frac{\pi}{2}vJ_{\mathrm{L}}J_{\mathrm{L}} + \frac{2\pi}{3}v\mathbf{J}_{\mathrm{L}}\cdot\mathbf{J}_{\mathrm{L}}$$

(and similarly for T_{R}). Here we have re-instated the velocity of light, v.

We now consider the effect of the Hubbard interaction, U, in the continuum limit. Writing the couplings in terms of $\psi_{\mathrm{L,R}}$ we get terms with four powers of ψ_{L} (or four powers of ψ_{R}) and also terms with two ψ_{L}'s and two ψ_{R}'s. Only four-fermi terms with no derivatives will be retained. The high-derivative terms are irrelevant at the free fermion fixed point. The completely left-moving (or completely right moving) terms can be written quadratically in the currents and thus simply renormalize the speed of light in (10). This renormalization is different for the terms

in (10) involving charge and spin currents. A theory with two different "velocities of light" could not, of course, be Lorentz-invariant. However, as we shall see, only the spin part of T will be retained in the low-energy theory so this problem will not arise.

The left-right terms correspond to Lorentz-invariant interaction terms in the Lagrangian (the same terms as in Hamiltonian, but with a change of sign.) Thus they can be treated using field theory methods. There are three different Lorentz-invariant interactions permitted by symmetries of the Hubbard model. We see from the definition of the continuum fermi fields, that the single charge and spin symmetries of the Hubbard model correspond to the diagonal subgroup of the chiral symmetries of the free fermion theory, under which left and right fermions transform the same way. The three interactions permitted by these diagonal symmetries are

$$L_{\mathrm{int}} = \frac{1}{4}\lambda_1 J_{\mathrm{L}}J_{\mathrm{R}} + \frac{1}{16}\lambda_2 \left[\left(\varepsilon_{\alpha\beta}\psi_{\mathrm{L}}^{\dagger\alpha}\psi_{\mathrm{L}}^{\dagger\beta}\right)\left(\psi_{R\gamma}\psi_{R\delta}\varepsilon^{\gamma\delta}\right) + \mathrm{h.c.}\right] + \lambda_3\mathbf{J}_{\mathrm{L}}\cdot\mathbf{J}_{\mathrm{R}}$$

Here $\varepsilon_{\alpha\beta}$ is the anti-symmetric tensor ($\varepsilon_{12} = 1$). These coupling constants have the values

$$-\lambda_1 = -\lambda_2 = \lambda_3 = \frac{U}{t}.$$

(L_{int} refers to the interaction part of the *Minkowski* Space Lagrangian.) At this point we bosonize. The most elegant way of doing this is to use Witten's non-abelian bosonization [23]. The version that is most useful in this case [24] is to rewrite the theory in terms of an SU(2) Wess–Zumino–Witten (WZW) matrix, $g(\mathbf{x})$ representing the SU(2) degrees of freedom, and a single scalar field, $\varphi(\mathbf{x})$ representing the $U(1)$ degrees of freedom. The free fermion theory is equivalent to decoupled theories for φ and g:

$$L(\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$$
$$S_{\mathrm{WZW}}(g) = \frac{1}{8\pi}\int \mathrm{d}^2x\ \mathrm{tr}\,\partial_\mu g^\dagger\partial^\mu g + \frac{1}{12\pi}\int \mathrm{d}^3x\,\varepsilon^{\mu\upsilon\lambda}\,\mathrm{tr}\,g^\dagger\partial_\mu g g^\dagger\partial_\upsilon g g^\dagger\partial_\lambda g.$$

The second (Wess–Zumino) term is defined by extending the two-dimensional space (or space-time) to a three-dimensional half-space

($x_3 < 0$) and $g(x_\mu)$ is an arbitrary extrapolation of the function defined on the two-dimensional space (at $x_3 = 0$) which approaches **1** at $x_3 \to -\infty$. The boundary value determines the Wess–Zumino term up to a term of the form $2\pi n$ (or $2\pi i n$ in Euclidean space) where n is an integer. Thus the path-integral is well defined if the coefficient in front of this term is an integer. (Here we have the lowest possible integer, $k = 1$.) The coefficient of the first term is then fixed by the condition of chiral symmetry or equivalently by the fixed point of the renormalization group. The currents are written:

$$J_{\mathrm{L}} = -\frac{1}{\sqrt{4\pi}}\partial_+\varphi, \qquad J_{\mathrm{R}} = \frac{1}{\sqrt{4\pi}}\partial_-\varphi$$
$$\mathbf{J}_{\mathrm{L}} = -\frac{i}{4\pi}\,\mathrm{tr}\,\partial_+ g g^\dagger \sigma, \quad \mathbf{J}_{\mathrm{R}} = \frac{i}{4\pi}\,\mathrm{tr}\,\partial_+ g^\dagger \partial_- g \boldsymbol{\sigma}$$

and the left-right fermion product as

$$\psi_{\mathrm{L}}^{+\alpha}\psi_{R\beta} \propto g_\beta^\alpha \exp\left(i\sqrt{2\pi}\varphi\right).$$

The λ_2 interaction is independent of g because it only involves the determinant of g which is one. This could have been predicted from the fact that this interaction is a chiral SU(2) singlet, and therefore cannot involve g. Thus the Lagrangian decouples into separate pieces involving φ or g only:

$$L(\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi\left(1 - \frac{\lambda_1}{8\pi}\right) + \text{constant} \cdot \lambda_2 \cdot \cos\sqrt{8\pi}\varphi$$
$$S(g) = S_{\mathrm{WZW}}(g) + \lambda_3 \int \mathrm{d}^2x\, \mathbf{J}_{\mathrm{L}} \cdot \mathbf{J}_{\mathrm{R}}\,.$$

Note that bosonization has allowed a complete separation of the charge and spin degrees of freedom in the continuum limit! The charge sector should be massive for any finite $U > 0$ since upon rescaling φ by $(1 - \lambda_1/8\pi)^{1/2}$ we get a relevant sine–Gordon interaction: $\cos[8\pi/(1 - \lambda_1/8\pi)^{1/2}\varphi$. Thus we see that the charge gap sets in at arbitrarily small U, as predicted by mean field theory. We will come back to this a bit later.

What of the spin sector? We have the $k = 1$ critical theory plus an additional interaction, λ_2. The interaction is marginal (has dimension 2); it will be marginally relevant or irrelevant depending on the sign

of the coupling. Let us calculate the renormalization group equation. Actually, let's do a slightly more general calculation than we need, for future use. In general the one-loop: β -function can be obtained from the operator product expansion. For a set of marginal interactions, O_i with (Minkowski space) Lagrangian:

$$L' = \sum_i \lambda_i O_i,$$

the: β -function:

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}\ln L} = \beta_{ijk}\lambda_j\lambda_k,$$

with:

$$O_i(\mathbf{x})O_j(\mathbf{y}) \to \beta_{ijk}\frac{O_k(\mathbf{x})}{\pi(\mathbf{x}-\mathbf{y})^2}.$$

This follows from expanding e^{-S} to quadratic order in L', using the operator product expansion and the integral:

$$\int \frac{\mathrm{d}^2\mathbf{x}}{\mathbf{x}^2} \approx 2\pi \ln L,$$

where L is the infra-red cut-off. We consider an anisotropic interaction:

$$L' = \sum_a \lambda_a J_{\mathrm{L}}^a J_{\mathrm{R}}^a.$$

(Don't confuse these λ_a's with the three coupling defined in the Hubbard model.) In general the operator product expansion for a current and *any* operator (of arbitrary dimension) is:

$$J_{\mathrm{R}}^a(z)O(z') = \frac{S_{\mathrm{R}}^a O(z')}{2\pi i(z - z')},$$

where S_{R}^a is the generator of $\mathrm{SU}(2)_{\mathrm{R}}$ transformations in the representation under which O transforms. (This follows from doing a contour integral over z which gives the commutator of the two operators.) Thus:

$$J_{\mathrm{R}}^a(z)J_{\mathrm{R}}^b(z') = \varepsilon^{abc}\frac{J_{\mathrm{R}}^c}{2\pi i(z - z')},$$
$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}\ln L} = -\frac{\lambda_2\lambda_3}{2\pi}$$
$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}\ln L} = -\frac{\lambda_1\lambda_3}{2\pi}$$
$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}\ln L} = -\frac{\lambda_2\lambda_1}{2\pi}$$

Let us now specialize to a general $U(1)$ invariant interaction:

$$L' = \frac{1}{2}\lambda_{+-}(J_L^+ J_R^- + J_L^- R_R^+) + \lambda_3 J_L^z J_R^z :$$

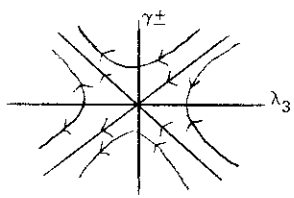
$$\frac{d\lambda_{+-}}{d\ln L} = -\frac{\lambda_{+-}\lambda_3}{2\pi}$$

$$\frac{d_3}{d\ln L} = -\frac{\lambda_{+-}^2}{2\pi}.$$

These are well-known *RG* equations which occur in a number of problems, including Kosterlitz's analysis [25] of the $2D$ classical XY model. Noting that

$$\frac{d\left(\lambda_{+-}^2 - \lambda_3^2\right)}{d\ln L} = 0,$$

we see that the *RG* flow lines are hyperbolas:



For $\lambda_3 > 0$ the couplings flow to the fixed line on the positive λ_3 axis, but for $\lambda_3 < 0$ they flow to infinity. Finally, let us consider the SU(2) invariant case, $\lambda_{+-} = \lambda_3$ which occurs in the Hubbard model. This condition determines the separatrix: the straight line *RG* flow to zero coupling. For λ initially positive, as in the positive U Hubbard model, it renormalizes to zero, and we conclude that the critical theory is simply the $k = 1$ WZW model.

Again we should ask whether higher order effects might destroy the massless behavior that we have found. Are there other operators which might be generated which could produce a mass for the spin excitations? The only Lorentz invariant relevant operators in the WZW models are polynomials in g. In the $k = 1$ case the only polynomial which exists as an independent operator in the quantum theory is g itself. This follows from the non-abelian bosonization equivalence. A second order polynomial would correspond to a fermionic operator which is zero by Fermi statistics.

$$g_{\{\beta}g_{\delta\}}^{\{\alpha\gamma\}} \exp\left(i\sqrt{8\pi}\varphi\right) \propto \psi_L^{+\{\alpha}\psi_{R\{\beta}\psi_L^{+\gamma\}}\psi_{R\delta\}} = 0.$$

[Here we symmetrize the indices since the anti-symmetric product gives $\det(g) = 1$.] The diagonal SU(2) symmetry of the Hubbard model would thus permit only the marginal $\mathbf{J}_L \cdot \mathbf{J}_R$, and the relevant $\mathrm{tr}\, g$, with dimension 1/2. Fortunately, there is a a Z_2 symmetry which again saves the day. This is the symmetry of translation by one site. This is a subgroup of the chiral SU(2) symmetry:

$$\psi_L \to i\psi_L, \quad \psi_R \to -i\psi_R.$$

In bosonized form it is $g \to -g$. (There appears to be a second discrete symmetry: $\varphi \to \varphi + \sqrt{\pi/2}$, which has the same effect on the spin or fermion variables. This actually implies that the fields g and φ are entwined by the identification of (g, φ) with $(-g, \varphi + \sqrt{\pi/2})$. The relevant $\cos\sqrt{8\pi}\varphi$ interaction in the charge sector leads to spontaneous breaking of the $\varphi \to \varphi + \sqrt{\pi/2}$ symmetry, so we must identify the one remaining unbroken discrete symmetry, $g \to -g$, with the translation symmetry of the lattice problem.)

Thus we conclude that the $k = 1$ WZW point could be a stable fixed point for the Hubbard model. It is then quite reasonable that the spin sector flows to this critical point for *all* U. Increasing U increases the mass in the charge sector. Integrating out the massive charge boson, φ, we obtain the effective Lagrangian for the spin degrees of freedom. However, the only operator permitted in this Lagrangian which is not irrelevant, is λ_3 (reverting to the notation introduced for the Hubbard model.) Thus all the charge sector can do is fix a value for this coupling. Noting that for small positive U, λ_3 is positive and increasing, with U, it is quite plausible that it remains positive for all $U > 0$. Assuming no intervening fixed points L_{eff} continues to renormalize to the $k = 1$ fixed point. Thus we identify the $k = 1$ WZW model with the critical theory for the isotropic $s = 1/2$ spin chain.

The correlation functions of the Hubbard or Heisenberg models can be read off. Since we have a relevant $\cos\beta\varphi$ interaction in the charge sector we expect $< e^{i\beta\varphi} > \neq 0$, i.e. breaking of the discrete symmetry,

$$\varphi \to \varphi + 2\pi/\beta.$$

The spectrum of the sine–Gordon model has been calculated exactly [26]. There is a finite mass gap and the spectrum contains a soliton, anti-soliton pair, reflecting the broken discrete symmetry. The φ propagator should decay exponentially. The spin correlation functions can be

expressed in terms of those of the WZW model:

$$\frac{\mathbf{S}(x)}{a} = (\mathbf{J}_{\mathrm{L}} + \mathbf{J}_{\mathrm{R}}) + (-1)^{x/a}\text{constant} \cdot \cos\sqrt{2\pi}\varphi \,\mathrm{tr}\, g\boldsymbol{\sigma}.$$

$\cos\sqrt{2\pi}\varphi$ may be replaced by its expectation value in calculating the long-range correlation function. g has dimension 1/2 so:

$$\left\langle S^a(x,t)S^b(0,0)\right\rangle \approx \delta^{ab}\left[-\frac{1}{8\pi^2}\left(\frac{1}{x_-^2}+\frac{1}{x_+^2}\right) + (-1)^{x/a}\text{const}\cdot\left(x^2-t^2\right)^{-1/2}\right].$$

A valuable check on this critical theory is to compare with the one obtained from the Jordan-Wigner transformation, which was essentially an extrapolation from $J_z = 0$. We concluded before that the isotropic point corresponded to radius $R = 1/\sqrt{2\pi}$ for the free boson. This gives the same critical exponents as the WZW model, and is, in fact, precisely equivalent. One way of seeing this is to apply *abelian* bosonization to the continuum limit of the Hubbard model. Thus we introduce 2 free boson, φ_1 and φ_2 to represent the 2 spin components ψ_1 and ψ_2. We can then identify the charge and spin bosons as:

$$\varphi = \frac{\varphi_1+\varphi_2}{\sqrt{2}}, \quad \varphi_s = \frac{\varphi_1-\varphi_2}{\sqrt{2}}.$$

This can be checked by noting that:

$$J_{\mathrm{R}} = :\psi_{\mathrm{R}}^{+\alpha}\psi_{R\alpha}: = \frac{1}{\sqrt{2\pi}}\partial_-\varphi,$$

$$J_{\mathrm{R}}^z = \frac{1}{\sqrt{8\pi}}\partial_-\varphi_s,$$

$$J_{\mathrm{R}}^- \propto :\exp\left(i\sqrt{8\pi}\varphi_{sR}\right):, \text{ etc.}$$

φ is the charge boson introduced earlier and φ_s is equivalent to the WZW field, g. Comparing bosonization formulas for $\psi_{\mathrm{L}}^{+\alpha}\psi_{R\beta}$ we conclude:

$$g_\beta^\alpha \propto \begin{bmatrix} \exp\left(i\sqrt{2\pi}\varphi_s\right) & \exp\left(i\sqrt{2\pi}\tilde{\varphi}_s\right) \\ \exp\left(-i\sqrt{2\pi}\tilde{\varphi}_s\right) & \exp\left(-i\sqrt{2\pi}\varphi_s\right)\end{bmatrix}$$

We could now consider a non-isotropic spin chain using non-abelian bosonization. Within the Hubbard model approach, we could add an explicit nearest neighbor spin-spin interaction:

$$H' = \frac{\Delta J_z}{4}\sum_i \left(\psi_i^\dagger\sigma^z\psi_i\right)\cdot\left(\psi_j^\dagger\sigma^z\psi_j\right).$$

Our previous discussion of symmetry and allowed operators implies that this can only generate an anisotropic current-current interaction in the WZW model:

$$L' = \frac{\lambda_{+-}}{2}\left(J_{\mathrm{L}}^+J_{\mathrm{R}}^- + J_{\mathrm{L}}^-J_{\mathrm{R}}^+\right) + \lambda_3 J_{\mathrm{L}}^z J_{\mathrm{R}}^z,$$

with $\lambda_3 - \lambda_{+-} \propto -\Delta J_z$.

Thus we move along the critical line $\lambda_3 > 0$ with increasing planar anisotropy ($\Delta J_z < 0$). This corresponds to rescaling the spin bosons radius, in the other approach. On the other hand Ising anisotropy ($\Delta J_z > 0$) drives the theory across the separatrix and produces a gap. We can use the *RG* equations to calculate how this gap scales with ΔJ_z. Initially,

$$\lambda_{+-}^2 - \lambda_3^2 = c(\Delta J_z),$$

where c is a constant. This condition is preserved along the *RG* trajectory. We now calculate the length scale ξ at which the effective couplings have passed near the origin and have moved away a distance of $O(1)$. Integrating the *RG* equations we find:

$$\xi \propto \exp\left[\text{constant}/\sqrt{\Delta J_z}\right] \propto m^{-1}.$$

The mass turns on exponentially slowly with anisotropy. In this phase we expect

$$\left\langle g_\beta^\alpha\right\rangle \propto (\sigma^z)_\beta^\alpha \Rightarrow \langle S_i^z\rangle \propto \pm 1.$$

There is a spontaneously broken Z_2 symmetry: $g \to \sigma_2 g \sigma_2$, or $\mathbf{S} \to (-S^x, S^y, -S^z)$. (There is no spontaneous breaking of *continuous* symmetries since the SU(2) symmetry is explicitly broken down to $U(1)\times Z_2$.)

We could also examine the effect of explicitly breaking the Z_2 symmetry, translation by one site, while preserving the SU(2) ie:

$$H \to J\sum_i \mathbf{S}_i\cdot\mathbf{S}_{i+1}\left[1+\alpha(-1)^i\right].$$

This breaks the $g \to -g$ symmetry of the WZW model, and permits the interaction $\alpha \operatorname{tr} g$. This is a relevant operator, of dimension 1/2, and produces a gap $\propto \alpha^{3/2}$.In φ_s language, we have added the interaction $\operatorname{tr} \sqrt{2\pi}\varphi_s$. We see that there must be a hidden SU(2) symmetry of the sine–Gordon model [26] at this value of β as well as at $\beta \to \sqrt{8\pi}$. At $\beta = \sqrt{2\pi}$ the spectrum consists of the soliton doublet and one breather (bound-state) which is degenerate with the doublet. Together they form a triplet, and it can be seen that the exact S-matrix has the corresponding SO(3) symmetry [27].

Another way of torturing the Heisenberg model is to add a second nearest neighbor coupling. This doesn't break any symmetries, so the model may remain massless. However, if the second nearest neighbor coupling $J_2 > 0$, then we find that the coupling constant λ_3 (i.e. the coefficient of $\mathbf{J}_\mathrm{L} \cdot \mathbf{J}_\mathrm{R}$) is reduced. Eventually it passes through zero and changes sign at some J_c. For negative λ_3 we expect the WZW model to be massive, with

$$m \propto \mathrm{e}^{-2\pi/\lambda_3} \propto \mathrm{e}^{-\mathrm{const}/(J_2 - J_c)}.$$

In the massive phase, where the current -current coupling grows large in the infra-red, we do not expect chiral symmetry to be present and

$$\left\langle g_\beta^\alpha \right\rangle \propto \delta_\beta^\alpha.$$

The Z_2 symmetry, $g \to -g$ is spontaneously broken. This is verified by the soliton spectrum of the sine–Gordon model at $\beta \to \sqrt{8\pi}$. (I will discuss some numerical results on the first and second nearest neighbor spin chain later.)

9. Higher spin, Bethe ansatz and finite-size scaling

So far, the discussion of critical behavior has been entirely about the $s = 1/2$ case. I would now like to generalize to higher s. Part of the motivation will be to understand the difference between s integer and half-integer, and the behavior of the $O(3)$ non-linear σ-model (not to be confused with the WZW models) for various values of θ. In all cases the critical behavior will be described by WZW models with various values of the topological coupling constant, k.

Before going into the details let us ask why these models should occur as the critical theories [20,22]. This can be understood by realizing

that any theory invariant under parity (as generally holds for lattice models) and a continuous symmetry group G, will necessarily have a chiral symmetry group (usually $G_\mathrm{L} \times G_\mathrm{R}$) at any possible critical point. The reason is simply that only chiral symmetries are compatible with conformal invariance.

To see this, suppose that we have some conformally invariant theory which has a continuous symmetry and an associated conserved current, J_μ. We may always define the left and right components:

$$J_\mathrm{R} = J_0 + J_1, \quad J_\mathrm{L} = J_0 - J_1.$$

These obey the conservation equation:

$$\partial_+ J_\mathrm{R} + \partial_- J_\mathrm{L} = 0.$$

The corresponding symmetry will be chiral if $\partial_+ J_\mathrm{R}$ and $\partial_- J_\mathrm{L}$ are separately 0. This will follow from the assumed conformal invariance and the fact that a conserved current must always have the canonical scaling dimension, 1. (This is essentially true because its integral is a pure number, a conserved charge.) Lorentz invariance implies that

$$\langle J_\mathrm{R}(\mathbf{x}) J_\mathrm{R}(\mathbf{0}) \rangle = \frac{f\left(\mathbf{x}^2\right)}{x_+^2},$$

where f is some scalar function. But the given scaling dimension of J implies that $f =$ constant. Thus the operator $\partial_+ J_\mathrm{R}$ has vanishing 2-point function. This implies that is it zero as an operator. Similarly for J_L. (I assumed parity invariance only to rule out the possibility that J_L or J_R was identically zero.)

Thus if we have some lattice model with a continuous symmetry group G which commutes with parity (is left-right symmetric), then *if* the model flows to a critical point, the symmetry must get enlarged to $G_\mathrm{L} \times G_\mathrm{R}$ at the critical point.

The WZW models represent the minimal critical theories with chiral symmetry. Given any conformally invariant theory with SU(2) currents, we may further classify the theory by the commutations relations obeyed by the currents. The most general possibility allowed by Lorentz invariance and dimensional analysis is the Kac–Moody algebra with central charge k:

$$\left[J_\mathrm{L}^a(x), J_\mathrm{L}^b(y)\right] = i\varepsilon^{abc} J_\mathrm{L}^c(x)\delta(x-y) + \frac{k}{2\pi}\delta^{ab}\delta'(x-y)$$

The central charge, k, must be integer valued in order to get a spectrum which is bounded below. We may define an energy-momentum tensor quadratic in currents. This is not necessarily the full energy-momentum tensor of the theory, but it at least correctly generates space-time translations of the currents themselves (this follows from the current commutation relations). It thus follows that the full energy-momentum tensor can be written as the current part plus an additional part which commutes with all the currents:

$$T_{\mathrm{L}} = T_{\mathrm{L}}^1 + T_{\mathrm{L}}^2$$

where T_{L}^1 is the term quadratic in currents and

$$\left[T_{\mathrm{L}}^2, T_{\mathrm{L}}^1\right] = \left[T_{\mathrm{L}}^2, J_{\mathrm{L}}\right] = 0.$$

Therefore the conformal anomaly parameter, c, will be the sum of that for T_{L}^1 and that for T_{L}^2. Thus the smallest possible value of c (and hence the fewest massless particles) arises when $T_{\mathrm{L}}^2 = 0$. Generally speaking we don't expect accidental massless particles apart from those required by some general principle, like chiral symmetry, so we would expect that the generic critical theory for a conformal system with some specified Kac–Moody algebra would be the corresponding WZW model. Examples of non-minimal models can easily be constructed by adding additional decoupled massless fields which are invariant under the symmetry group. In this case all operators are simply products of operators in the WZW theory and operators in the decoupled theory. Less trivial cases may also exist. One such example is the free fermion theory itself. The energy momentum tensor for a system of $n_c n_f$ free fermions can be written in Sugawara form using the $\mathrm{SU}(n_c)$, $\mathrm{SU}(n_f)$ and $U(1)$ currents, only. However, local expressions for fermion operators in terms of the corresponding WZW fields do not exist, in general.

We begin with the standard spin-s antiferromagnetic Heisenberg model:

$$H = J \sum \mathbf{S}_n \cdot \mathbf{S}_{n+1}, \quad \mathbf{S}_n^2 = s(s+1) \tag{11}$$

We find it convenient to use a generalized Hubbard model representation for the spin chain. The model actually mimics the way spins are produced from electrons in real crystals. The best higher spin antiferromagnets [like $(CH_3)_4NMnCl_3$] with $s = 5/2$ have transition metals as the magnetic ions, with half-filled outer d-shells [like Mn^{++}]. A strong Hund's rule coupling between the five electrons in the d-shell of each ion

makes the spins line up, forming an $s = 5/2$ spin variable. The weaker exchange forces between electrons of *neighboring* ions then produce the antiferromagnetic interaction between the spins. In our model we will introduce $2s$ identical half-filled orbitals per lattice site to obtain spin s. The Hamiltonian contains a nearest neighbor hopping term and a Hund's rule coupling which is taken to infinity. We will refer to the orbital index, $i = 1, 2, \ldots, n_c$ as the *color*. The number of colors, $n_c = 2s$. $\psi_{i\alpha n}$ annihilates an electron of color i, spin α (equal $\pm 1/2$) at site n. Repeated spin or color indices are implicitly summed over. Thus the spin variables are written:

$$\mathbf{S}_n = \sum \mathbf{S}_{in} = \frac{1}{2} \psi_n^{+i\alpha} \boldsymbol{\sigma}_\beta^\alpha \psi_{i\beta n} \tag{12}$$

where $\mathbf{S}_{in}$ is the spin of the color i electron and the $\sigma^{a\prime}$ s are Pauli matrices. The Hund's rule coupling is

$$H_{\mathrm{HR}} = -U \sum \mathbf{S}_n^2$$

Using the identity

$$\boldsymbol{\sigma}_\beta^\alpha \cdot \boldsymbol{\sigma}_\varepsilon^\gamma = 2\delta_\varepsilon^\alpha \delta_\beta^\gamma - \delta_\beta^\alpha \delta_\varepsilon^\gamma .$$

this can be written

$$H_{\mathrm{HR}} = \frac{U}{4} \sum [2\left(\psi_n^{+i\alpha} \psi_{j\alpha n} - \delta_j^i\right) \cdot \left(\psi_n^{+j\beta} \psi_{i\beta n} - \delta_i^j\right) + \left(\psi_n^{+i\alpha} \psi_{i\alpha n} - n_c\right)^2]$$

Here we have dropped constants and terms proportional to the (conserved) total electron number. It can be seen that the ground-states of H_{HR} are all states with n_c electrons per site in a color singlet state (i.e. antisymmetric with respect to color). Such states have spin $s = n_c/2$. The hopping term is:

$$H_0 = -t \sum \left[\psi_n^{+i\alpha} \psi_{i\alpha,n+1} + \mathrm{h.c.}\right]$$

where h.c. stands for hermitian conjugate. Note that the full Hamiltonian is invariant under an $\mathrm{SU}(n_c)$ color symmetry and that the Hund's rule coupling favors local color singlet states. This is very reminiscent of Quantum Chromodynamics where the Hund's rule coupling results effectively from exchange of gluons (this leads however to a long-range

force, not a local one). The low-energy states of the Hubbard model correspond to baryons (color singlets formed antisymmetrically out of quarks) with residual spin interactions. In the limit $U \to \infty$, the Hamiltonian, projected onto the low-energy states, becomes simply the Heisenberg Hamiltonian. This can be seen by considering perturbation theory in the hopping term. Only second-order perturbation theory survives as $U \to \infty$. The intermediate state has $s = (n_c \pm 1)/2$ on two sites and hence excitation energy, $\Delta E = U/2$. The effective Heisenberg coupling is $J = 4t^2/nU$. This representation can be generalized to any dimension.

We will extract the critical theory for the antiferromagnetic chain out of this representation by considering first the case of small $U(U << t)$. The motivation is the same as in the case of the ordinary Hubbard model. We will search for a possible gapless low-energy sector of the theory at small U. The remaining states will have a gap that grows with U [being of order $\exp(-\text{constant} \cdot t/U)$ at small U]. The gapless sector is decoupled from the other states. Therefore the effect of taking $U \to \infty$ is simply to give these other states infinite energy.

The free theory gives us a theory of free Dirac fermions, as usual, this time with charge, spin *and* color degrees of freedom. There are corresponding chiral charge, spin and color symmetries.

Corresponding to these symmetries we have conserved currents. Introducing the generators of $SU(n_c)$ $T_i^{Aj} (A = 1, 2, 3, \ldots, n_c^2 - 1; \mathrm{tr}\, T^A T^B = (1/2)\delta^{AB}$; summation of repeated "A" indices will be implied) the light-cone components of the currents can be written:

$$
\begin{aligned}
J_{\mathrm{L,R}} &\equiv : \psi^{+i\alpha}_{\mathrm{L,R}} \psi_{\mathrm{L,R}i\alpha} :, \\
bf J_{\mathrm{L,R}} &\equiv \psi^{+i\alpha}_{\mathrm{L,R}} (1/2) \boldsymbol{\sigma}^{\beta}_{\alpha} \psi_{\mathrm{L,R}i\beta}, \\
J^A_{LR} &\equiv \psi^{+i\alpha}_{\mathrm{L,R}} T^{Aj}_i \psi_{\mathrm{L,R}j\alpha}.
\end{aligned}
$$

The energy-momentum tensor can be written in a form quadratic in the charge, spin and color currents (See [20] and references there in):

$$
T_{\mathrm{L}} = \frac{\pi}{2n_c} v J_{\mathrm{L}} J_{\mathrm{L}} + \frac{2\pi}{n_c + 2} v \mathbf{J}_{\mathrm{L}} \cdot \mathbf{J}_{\mathrm{L}} + \frac{2\pi}{n_c + 2} v J^A_{\mathrm{L}} J^A_{\mathrm{L}} \qquad (13)
$$

(and similarly for T_{R}).

We now consider the effect of a small Hund's rule coupling $U << t$. We will focus on its effect on the low-energy, relativistic sector of the Hilbert Space. Writing the Hund's Rule couplings in terms of $\psi_{\mathrm{L,R}}$ we get terms with four powers of ψ_{L} (or four powers of ψ_{R}) and also terms

with two ψ_{L}'s and two ψ_{R}'s. Only four-fermi terms with no derivatives will be retained. The higher-derivative terms are irrelevant at the free fermion fixed point and also at the non-trivial fixed points which we will encounter later. The completely left-moving (or completely right-moving) terms can be written quadratically in the currents and thus simply renormalize the speed of light in (13). This renormalization is different for the terms in (13) involving charge, spin and color currents. A theory with three different velocity of lights could not, of course, be Lorentz-invariant. However, as we shall see, only the spin part of T_{L} will be retained in the low-energy theory so only the spin velocity parameter will be relevant.

The left-right terms correspond to Lorentz-invariant interaction terms in the Lagrangian. Thus they can be treated using field theory methods. There are actually six different Lorentz-invariant interactions permitted by the symmetries of the Hubbard model. The single charge, spin and color symmetries of the Hubbard model correspond to the diagonal subgroup of the chiral symmetries of the free field theory, under which left and right fermions transform the same way. The six interactions permitted by these diagonal symmetries are

$$
\begin{aligned}
L_{\mathrm{int}} =& \lambda_1 J_{\mathrm{L}} J_{\mathrm{R}} + \lambda_2 \mathbf{J}_{\mathrm{L}} \cdot \mathbf{J}_{\mathrm{R}} + \lambda_3 J^A_{\mathrm{L}} J^A_{\mathrm{R}} \\
&+ \lambda_4 \left(\psi^{+i\alpha}_{\mathrm{L}} T^{Aj}_i \boldsymbol{\sigma}^{\beta}_{\alpha} \psi_{\mathrm{L}i\beta} \right) \cdot \left(\psi^{+k\gamma}_{\mathrm{R}} T^{Al}_k \boldsymbol{\sigma}^{\delta}_{\gamma} \psi_{\mathrm{R}l\delta} \right) \\
&+ \lambda_5 [\left(\varepsilon_{\alpha\beta} \psi^{+i\alpha}_{\mathrm{L}} \psi^{+j\beta}_{\mathrm{L}} \right) \left(\psi_{\mathrm{R}i\gamma} \psi_{\mathrm{R}j\delta} \varepsilon^{\gamma\delta} \right) + \mathrm{h.c.}] \\
&+ \lambda_6 \left(\psi^{+i\{\alpha}_{\mathrm{L}} \psi^{+j\beta\}}_{\mathrm{L}} \cdot \psi_{\mathrm{R}i\{\alpha} \psi_{\mathrm{R}j\beta\}} + \mathrm{h.c.} \right)
\end{aligned}
$$

Here $\varepsilon_{\alpha\beta}$ is the anti-symmetric tensor ($\varepsilon_{12} = 1$) and the curly brackets indicate symmetrization of the spin indices. Due to Fermi statistics the fifth term which is anti-symmetric in (left and right) spin indices is symmetric in color indices and conversely for the sixth term. These coupling constants have the values (with $v = 1$)

$$
\lambda_1 = \frac{3aU}{4n_c}, \quad \lambda_2 = \left(2 + \frac{1}{n_c} \right) aU, \quad \lambda_3 = -\frac{3aU}{2},
$$
$$
\lambda_4 = 8aU, \quad \lambda_5 = \frac{3aU}{8}, \quad \lambda_6 = -\frac{aU}{4}.
$$

We will be interested in more general Hamiltonians than the basic Heisenberg model of (11). In particular we may wish to allow arbitrary polynomial nearest neighbor interactions as well as second (and

possibly higher) nearest neighbor interactions:

$$H = \sum \left[P_1 \left(\mathbf{S}_n \cdot \mathbf{S}_{n+1} \right) + P_2 \left(\mathbf{S}_n \cdot \mathbf{S}_{n+2} \right) + \cdots \right].$$

The arbitrary functions P_1, P_2 can be assumed without loss of generality to be polynomials of degree $2s$. These can be incorporated into the Hubbard model formulation by simply making the replacement (12) for the spins in all terms beyond the basic Heisenberg Hamiltonian. Taking all these additional interactions to be small we may treat them perturbatively. They have the effect of renormalizing the speed(s) of light and the coupling constants, λ_1 to λ_6. Operators of higher order than quartic in the fermion fields are also generated. However these are irrelevant at the (unstable) free fermion fixed point. We *will* effectively consider them later after locating the non-trivial fixed points. For example, a bilinear nearest neighbor exchange term

$$\delta H = J_b \sum \left(\mathbf{S}_n \cdot \mathbf{S}_{n+1} \right)^2,$$

gives, in the continuum limit,

$$\begin{aligned} \delta H =& (J_b) \{ \left[\left(\mathbf{J}_{\mathrm{L}} + \mathbf{J}_{\mathrm{R}} \right)^2 - \left(\psi_{\mathrm{L}}^{+i\alpha} \sigma_\alpha^\beta \psi_{\mathrm{R}i\beta} + \mathrm{h.c.} \right)^2 \right]^2 \\ &+ \left[\left(\mathbf{J}_{\mathrm{L}} + \mathbf{J}_{\mathrm{R}} \right) \cdot \left(\psi_{\mathrm{L}}^{+i\alpha} \boldsymbol{\sigma}_\alpha^\beta \psi_{\mathrm{R}i\beta} + \mathrm{h.c.} \right) \right]^2 \} \end{aligned}$$

Wick contracting two pairs of fermion fields gives operators quartic in the Fermion fields. (Using the lattice propagator this calculation is ultraviolet finite.) Thus in general we should consider the six coupling constants, λ_i, to be arbitrary parameters.

The renormalization group equations now have many more terms than for $n_c = 1$ since six couplings are present. Furthermore the fourth and sixth operators are non-singlets under all three chiral groups so that there is no separation of the theory into sectors in general. Let us first consider the case where λ_4 and λ_6 are zero. (This could presumably be achieved by adjusting two parameters in the spin chain Hamiltonian such as biquadratic and second nearest neighbor couplings.) In this case the fifth term couples together the charge and color degrees of freedom but is a chiral *spin* singlet. Thus the renormalization group equation for λ_2 separates from that for the other three couplings. Again λ_2 flows to zero assuming it is initially positive, whereas the other three couplings

flow to large values. This suggests the existence of a massless decoupled spin sector as for $s = 1/2$. But what is the critical theory in this case?

To answer this question let us review a few features of the free fermion theory first. The structure of the energy-momentum tensor in (12) suggests that the theory can be separated into charge, spin and color sectors. Let us consider the commutation relations obeyed by the currents. We find [21] that the SU(2) currents obey the Kac–Moody algebra with central charge $k = n_c$. This is the algebra obeyed by the currents in the WZW model with topological coupling constant k. Furthermore, this model has an energy-momentum tensor quadratic in currents, as in (12). Indeed this WZW model represents the minimal conformal theory for a given value of k as discussed above. Likewise, the SU(n_c) currents obey the $k = 2$ Kac–Moody algebra and the $U(1)$ currents obey the usual Schwinger algebra. Let us consider a possible bosonization formula for the fermion bilinear, $\psi_{\mathrm{L}}^{\alpha i +} \psi_{j\beta}$. The only formula which has the right commutation relations with all three currents is:

$$\psi_{\mathrm{L}}^{\alpha i +} \psi_{j\beta} \propto g_\beta^\alpha h_j^i \exp i \frac{\sqrt{2\pi}}{n_c} \varphi,$$

where g is a $k = n_c$ SU(2) WZW field, h is a $k = 2$ SU(n_c) WZW field and φ is a free boson. The prefactor $\sqrt{2\pi/n_c}$ is fixed by the commutator with $J_{\mathrm{L}}, J_{\mathrm{R}}$. As a first check on the validity of this formula, consider the two point function of this operator. The dependence on the indices is correctly reproduced and the scaling dimension [24] is the total dimension of the spin, color and charge boson fields:

$$x = \frac{3}{2\left(2 + n_c\right)} + \frac{\left(n_c^2 - 1\right)}{n_c \left(n_c + 2\right)} + \frac{1}{2n_c} = 1.$$

However, it is easy to see that higher correlation functions are not correctly reproduced. In terms of the operator product expansion, it appears that for each composite fermion operator there corresponds a product of boson operators of the right symmetry and dimension, but the reverse is not true; i.e. there are too many boson operators. The free fermion theory may represent some sort of projection on the product of the three boson theories. This question probably deserves more investigation.

In any event, it appears likely that the low energy effective action for a fermion theory with interactions *which preserve the* SU*(2) chiral*

symmetry, is the $k = n_c$ SU(2) WZW model. In the simpler case of a chiral $U(1)$ symmetry, the existence of the chiral conserved currents implies the existence of a massless free boson: [28] i.e.

$$\partial^\mu J_\mu = 0 \Rightarrow J_\mu = \varepsilon_{\mu\nu}\partial^\nu \varphi,$$
$$\varepsilon^{\mu\nu}\partial_\mu J_\nu = 0 \Rightarrow \partial^2 \varphi = 0.$$

We expect a similar result to hold also in the non-abelian case. Given the existence of conserved chiral currents obeying the Kac–Moody algebra with some value of k, the corresponding minimal theory is the WZW model, as argued above. Thus, such an interacting theory would in general have a massive sector plus a decoupled massless WZW theory. However, any interactions which break the chiral symmetry would be expected, in general, to produce a mass for the WZW sector.

Returning to our generalized Hubbard model, we see that in the chiral SU(2) invariant case, $\lambda_4 = \lambda_6 = 0$, the low-energy effective action should be the $k = n_c$ WZW model. However, even an infinitesimal λ_4 or λ_6 should destabilize this fixed point. In terms of the spin-s Heisenberg models, it is thus plausible that there are certain multicritical points in the space of all Hamiltonians which are in the $k = 2s$ universality class. It has been established that the Bethe ansatz integrable spin-s Hamiltonians are of this type. These have polynomial nearest neighbor interactions: [29]

$$H = \sum_{nj} a_j P^j \left(\mathbf{S}_n + \mathbf{S}_{n+1}\right),$$

where P^j projects out spin-j, and the numerical coefficients, a_j are given by:

$$a_j = \sum_{i=1}^{j} \frac{1}{i}.$$

As discussed in Cardy's lectures in this series, the universality class can be determined from the conformal anomaly parameter, c, which controls the finite size scaling of the ground state energy, or equivalently, the low-temperature specific heat. Essentially, the specific heat is obtained from the two-point function of the Hamiltonian density, H:

$$C(T) = \int \mathrm{d}x < H(x)H(0) > /T^2.$$

This two-point function at finite T can be related to the conformal anomaly parameter, c. $C(T)$ was calculated exactly from the Bethe ansatz [29] (in terms of a one-dimensional integral) and appears to agree exactly with the value for the $k = 2s$ WZW model:

$$c = 3k/(2+k).$$

A second check was obtained from the $T = 0$ susceptibility: [30]

$$\chi = \sum_i \langle S_0^z S_i^z \rangle /T = \int \mathrm{d}x \, \langle J_0^z(x) J_0^z(0) \rangle /T.$$

This correlation function can be related to the Kac–Moody central charge, k, and the exact Bethe ansatz calculation verifies that $k = 2s$, once again. More recently the finite size scaling of the ground and excited state energies have been calculated using the Bethe ansatz for $s = 1/2$, 1 and 3/2 and again verify this identification [31,32]. This finite size scaling converges very slowly due to the presence of the marginal operator $g\ \mathbf{J}_\mathrm{L} \cdot \mathbf{J}_\mathrm{R}$ in the effective action. [See Cardy's lectures in this series for a discussion of finite-size scaling and conformal field theory.] Asymptotically, energy gaps scale with length L like $(2\pi v x/L)$ where v is the velocity of "light" and x the scaling dimension of the operator corresponding to the excited state. However, with a marginal operator present, there is a correction to this scaling of the form:

$$x \to x + \mathrm{const} \cdot g(L).$$

Here $g(L)$, the effective coupling constant, goes to zero as $1/\ln L$. This result is obtained by doing first order perturbation theory in g and then "renormalization group improving" the answer. Thus the constant in the above expression is the matrix element of the marginal operator in the excited state. For the operator $\mathbf{J}_\mathrm{L} \cdot \mathbf{J}_\mathrm{R}$, the matrix element is $\propto \mathbf{S}_\mathrm{L} \cdot \mathbf{S}_\mathrm{R}$, where $\mathbf{S}_\mathrm{L}, \mathbf{S}_\mathrm{R}$ are the left and right spin of the state. (All states in the Kac–Moody conformal towers of the WZW model can be classified by their chiral SU(2) quantum numbers.) For $g = 0$ there is a supermultiplet degeneracy of states of given $s_\mathrm{L}, s_\mathrm{R}$. For non-zero g, the chiral SU(2) is broken down to the diagonal subgroup, $\mathbf{S} = \mathbf{S}_\mathrm{L} + \mathbf{S}_R$, so the supermultiplets split up into multiplets of definite $\mathbf{S}$. These splittings are proportional to $g(L)$. The ground state energy contains a term $-\pi c/6L$. This also receives a correction from perturbation theory in g:

$$c \to c + \mathrm{constant} \cdot g(L)^3.$$

These predictions can be compared with *analytic* Bethe ansatz calculations [31]. An analytic asymptotic formula has been derived from the Bethe ansatz for the $s = 1/2$ chain, for the gap to the lowest energy state of given spin. The terms of $O(1/L)$ and $O(1/L \ln L)$ agree exactly with the field theory predictions from the $k = 1$ WZW model. [A similar calculation has been done for the ground state energy. While the $O(1/L)$ term agrees, there is a 10% discrepancy in the $O(1/L(\ln L)^3)$ term.] The field theory predictions can also be compared with *numerical* Bethe ansatz results, which are available [32] for $L \leq 2048$, 256 and 100 for the integrable models with $s = 1/2$, 1 and 3/2 respectively. The ground state energy and gap to the lowest singlet and triplet have been calculated for each model. This allows three independent determinations of $g(L)$ which we refer to as g_c, g_s and g_t respectively. Asymptotically, at large L, $g(L)$ should go to zero logarithmically and the three estimates of g should agree up to $O(g^2)$ corrections. The numerical results are shown in Fig. 1.

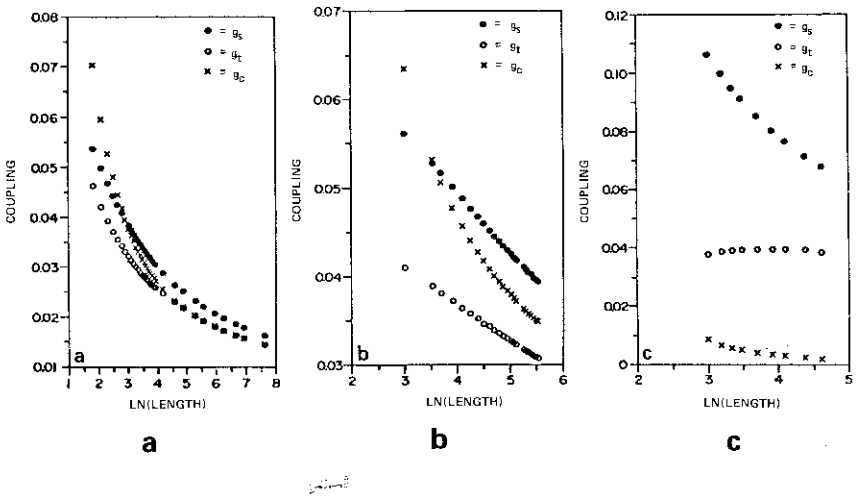


Fig. 1. Effective coupling vs. length as estimated from the triplet gap, singlet gap and ground state energy, for the integrable models for a) $s = 1/2$, b) $s = 1$ and c) $s = 3/2$.

Note that exponentially large chains are necessary in order for $g(L)$ to become small, in general. Another approach to finite size scaling, is to adjust a parameter in the spin Hamiltonian so that the bare value of g may be tuned to zero. Then excellent finite size scaling can be

obtained for chains of much more modest length. For the nearest plus next nearest neighbor $s = 1/2$ model the marginal coupling, g, vanishes for $J_2/J_1 \approx .25$. Low-lying excitation energies for $p \approx 0$ and π are shown in Fig. 2 for $J_2/J_1 = .25$ and 0.

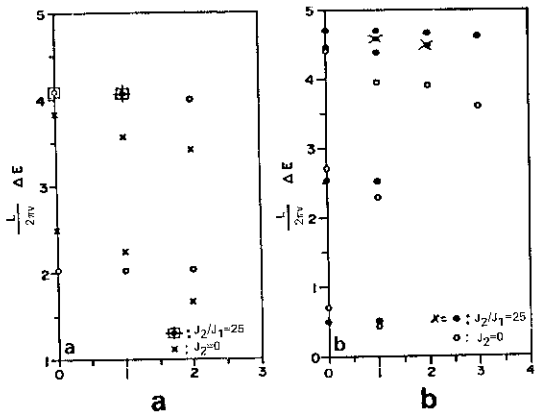


Fig. 2. Low energy states of $s = 1/2$ models with nearest neighbor coupling only and with first plus second nearest neighbor couplings with $J_2/J_1 = .25$: a) $p = 0$, b) $p = \pi$.

In the former case the supermultiplet structure immerges very clearly for $L = 20$. The degeneracies are precisely those predicted by the $k = 1$ WZW model.

These checks against the Bethe ansatz validate the mapping of the integrable models onto the $k = 2s$ WZW model, and of general $s = 1/2$ Hamiltonians onto the $k = 1$ WZW model. What happens for more general Hamiltonians and $s > 1/2$? It is convenient to consider Hamiltonians which are close to the integrable one. We can then represent the low-energy theory by the $k = 2s$ WZW model with relevant operators added. In general we should expect all relevant operators permitted by symmetry to appear. Apart from the marginal current-current interaction, there are additional relevant operators that respect diagonal SU(2) and the $g \to -g$ inversion symmetry. These are primary fields of the Kac–Moody and conformal algebra. They are classified by their spin quantum numbers under the left and right SU(2)'s. Primary fields [33] exist with equal left and right spins, j, equal to an arbitrary

half-integer or integer such that $j \leq k/2$; their scaling dimension is $x = 2j(j+1)/(2+k)$.

They correspond to polynomials in the fundamental field, g, of the WZW model:

$$g^{\alpha_1}_{\beta_1} g^{\alpha_2}_{\beta_2} g^{\alpha_3}_{\beta_3} \cdots,$$

where the upper indices are symmetrized (and hence so are the lower ones). The restriction on the maximum value of j can be obtained from the free fermion theory. We can construct operators out of $2j\psi^\dagger_{\rm L}$'s and $2j\psi_{\rm R}$'s with left and right spin j. But these operators are zero due to fermi statistics for $2j > n_c$. Only the integer j fields respect the inversion symmetry. Therefore there is a relevant operator respecting all symmetries of the generalized Hubbard model for each positive integer $j \leq k/2$ whose dimension $2j(j+1)/(2+k) < 2$. For large k the number of relevant operators, $n_{\rm R}$, grows like $\sqrt{k}$.

Thus we should expect that if we adjust a number of parameters in the spin chain Hamiltonian equal to the number of relevant operators, then the critical theory will be the $k = 2s$ WZW model. From this point of view, we see that the integrable Hamiltonians correspond to such special multi-critical points. Why the integrable models should happen to be multi-critical rather than generic remains an interesting and unanswered question. However since the number of possible nearest neighbor couplings is $2s = k$ we see that the integrable Hamiltonians are not the *only* such multi-critical nearest neighbor Hamiltonians but should lie in a space of dimension $2s - n_{\rm R}$.

One might expect that the theories would develop a gap for generic Hamiltonians with $s > 1/2$. After all, it was chiral symmetry which was protecting a massless sector for $s = 1/2$ and at the multi-critical points for general s. If this symmetry is broken by relevant operators why should there be *any* massless sector? The answer to this question is quite surprising. There appears to be a type of topological stability which protects a massless sector for odd n_c (half-integer s) whereas for even n_c (integer s) there is in general a gap. To understand this, let's consider the case of very large n_c. We will imagine that the relevant chiral symmetry breaking operators are small so that the system is flowing away from the $k = n_c$ fixed point. At large k, the WZW model becomes essentially a free theory, so a semi-classical analysis becomes justified. That is, writing $g = \exp(i\boldsymbol{\sigma} \cdot \boldsymbol{\varphi}/2)$, we may regard the triplet of bosons, $\boldsymbol{\varphi}$, as being essentially free fields. Let us consider only the most relevant

interaction, which can be written as

$$L_{\rm int} = -V = -\lambda(\mathrm{tr}\, g)^2$$

where V is the potential energy density. Let us analyze classically the effect of this operator. If $\lambda < 0$, then the minimum of V occurs at $g = \pm\mathbf{1}$ ($\boldsymbol{\varphi} = 0$ or $|\boldsymbol{\varphi}| = 2\pi$). Thus the discrete symmetry $g \to -g$ is spontaneously broken. Correspondingly, the spin chain is in a spontaneously dimerized phase with the symmetry of translation by one site spontaneously broken. Expanding V in powers of $\boldsymbol{\varphi}$ we get

$$V \approx \text{constant} + |\lambda|\boldsymbol{\varphi}^2 + \cdot.$$

Thus all three degrees of freedom obtain a mass $\propto \sqrt{\lambda}$. The spontaneously dimerized phase has a mass gap.

On the other hand, if $\lambda > 0$, then the minimum of V occurs at $|\boldsymbol{\varphi}| = \pi$, $g = i\boldsymbol{\sigma} \cdot \hat{\varphi}$ where $\hat{\boldsymbol{\varphi}} = \boldsymbol{\varphi}/|\boldsymbol{\varphi}|$. Note that diagonal SU(2) rotations:

$$g^\beta_\alpha \to U^\gamma_\alpha g^\delta_\gamma U^{+\beta}_\delta$$

transform $\boldsymbol{\varphi}$ by the corresponding SO(3) rotation:

$$\varphi^i \to R^i_j \varphi^j.$$

The minimum of V now occurs at an arbitrary point on the sphere and breaks the diagonal SU(2) symmetry. Correspondingly the rotational symmetry of the Hubbard model or spin chain is spontaneously broken, in this approximation. The longitudinal component of $\boldsymbol{\varphi}$ obtains a mass $\propto \sqrt{\lambda}$ but the transverse components (the two angles parametrizing $\hat{\boldsymbol{\varphi}}$) are massless Goldstone bosons in this semi-classical approximation. Let us go beyond the semi-classical approximation by considering the interactions between these would-be Goldstone bosons. Thus we integrate out the longitudinal part of $\boldsymbol{\varphi}$. To lowest order we simply replace g by $i\boldsymbol{\sigma} \cdot \hat{\varphi}$ in the WZW Lagrangian. The first term in $S_{\rm WZW}$ gives

$$L = \frac{k}{4\pi} \partial_\mu \hat{\boldsymbol{\varphi}} \partial^\mu \hat{\boldsymbol{\varphi}}.$$

This is the Lagrangian of the $O(3)$ σ-model with coupling constant $2\pi/k << 1$. The interactions between these would-be Goldstone bosons

as represented by the $O(3)$ σ-model, are expected to restore the symmetry and produce an exponentially small mass gap in $(1+1)$ dimensions

$$m \propto \exp(-k).$$

Thus we expect that the transverse components will not be strictly massless for any finite k although they will be much lighter than the longitudinal component.

However this analysis is not yet complete, because we have neglected the effect of the Wess–Zumino term. It also makes a relevant contribution after integrating out the longitudinal fluctuations of $\boldsymbol{\varphi}$. In fact freezing the magnitude of $\boldsymbol{\varphi}$ reduces the π_3 topological term of the WZW model to the π_2 topological term of the $O(3)$ σ-model. A simple way of seeing this is to write an extrapolation of the field g to the three dimensional half-space as

$$g(x_\mu) = \exp\left[if(x_3)\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\varphi}}(x_1,x_2)\right]$$

where f is any smooth function such that $f(-\infty)=0$ and $f(0)=\pi/2$. The Wess–Zumino term then becomes

$$\Gamma = 4iQ\int \mathrm{d}x_3 \frac{\mathrm{d}f}{\mathrm{d}x_3}\sin^2 f = i\pi Q$$

where f is the integer-valued π_2 topological term,

$$Q \equiv \frac{1}{8\pi}\int \mathrm{d}^2x\,\hat{\boldsymbol{\varphi}}\cdot(\partial_\mu\hat{\boldsymbol{\varphi}}\times\partial_v\hat{\boldsymbol{\varphi}})\,\varepsilon^{\mu v}.$$

(The normalization can be checked by the following argument. A simple soliton-like configuration which has a π_3 topological charge of one can be obtained by taking $\hat{\boldsymbol{\varphi}}$ to be an instanton with $Q=1$, and $f(x_3)$ a smooth function that increases monotonically from 0 at $x_3=-\infty$ to $\pi/2$ at $x_3=0$ to π at $x_3=\infty$. Thus $g(x_3=-\infty)=\mathbf{1}$, $g(x_3=0)=i\boldsymbol{\sigma}$ and $g(x_3=\infty)=-\mathbf{1}$. The Wess–Zumino term is normalized to be $2\pi i$ times the half-space integral of the π_3 topological charge density, in order that the ambiguity in Γ be $2\pi i$. But the half-space contains half the π_3 topological charge in this particular case so $\Gamma = i\pi$ for an instanton with $Q=1$.)

Thus we conclude that integer and half-integer spin chains may have different behavior in general (although for integrable Hamiltonians the behavior is essentially the same). In the integer-s case, our earlier discussion of the $\theta=0$ $O(3)$ σ-model makes it plausible that there is a finite mass gap away from the multicritical points. But what of the 1/2-integer case? We are back to the issue of understanding the $O(3)$ σ-model at $\theta=\pi$.

To settle this question let's go back and consider the simplest half-integer case, $s=1/2$. I argued earlier that we can apparently obtain the $O(3)$ σ-model even in this case with a large *ferromagnetic* second nearest neighbor interaction. The effective coupling is

$$g = \frac{2}{s(1+4J_2/J_1)^{1/2}},$$

which can be made small even for $s=1/2$. The large value of J_2 forces the disorder to occur at long wavelengths, since a one-dimensional ferromagnet has an ordered ground state. But the effect of a ferromagnetic J_2 is simply to increase the marginally irrelevant current-current coupling in the $k=1$ WZW model. Assuming no intervening fixed points this coupling will continue to flow to zero. Thus, we hypothesize that the $O(3)$ σ-model at $\theta=\pi$ has a massless sector, described by the $k=1$ WZW model. Note that the $O(3)$ σ-model at $\theta=\pi$ is not, itself, conformally invariant; it has a non-trivial beta function. Associated with this is a "dimensional transmutation" mass scale. The full spectrum of the theory may well contain some massive particles as well as the decoupled massless sector described by the WZW model. It is the non-existence of relevant operators which stabilizes this massless sector. Integrating out the massive modes can only renormalize the marginal coupling constant.

Returning to our analysis of large, odd k WZW models with relevant operators added, we now see that the action renormalizes from the higher odd k fixed point to the $k=1$ fixed point, in the phase of unbroken Z_2 symmetry ($\lambda>0$). We should really not restrict ourselves to a single relevant interaction but include all relevant operators permitted by symmetry. The classical potential can in general be written as a periodic function of $|\boldsymbol{\varphi}|$ only, due to the diagonal SU(2) symmetry. It is also symmetric under $|\boldsymbol{\varphi}|\to 2\pi-|\boldsymbol{\varphi}|$ which corresponds to $g\to -g$. For some range of parameters we expect this discrete symmetry to be unbroken and the minimum to lie at $|\boldsymbol{\varphi}|=\pi$. In this case the low-energy theory will be the $O(3)$ σ-model as above. For odd k we then expect cross-over to the $k=1$ fixed point. For another range of parameters, this discrete symmetry will be spontaneously broken and the system will

generically develop a gap. This corresponds to a short-range dimerized phase of the quantum spin chain. It now seems very likely that cross-over to the $k = 1$ fixed point occurs for *any* odd k. The only role of large k in the above arguments was to make the transverse components of φ much lighter than the longitudinal component. The numerical simulations [34] indicate that the above picture is correct for $k = 3$.

We are thus led to the conclusion that in a large class of models there is a massless phase described by the $k = 1$ WZW model. These models include half-integer spin chains, generalized Hubbard (or, in the continuum limit) Thirring models with an odd number of colors, WZW models with odd k and the $O(3)$ σ-model with $\theta = \pi$. It is not surprising that this should be the generic critical theory for SU(2) invariant systems since it is the *only* WZW model in which there are no relevant operators.

We expect the phase diagram for a half-integer-s quantum spin chain to have a massless region in the $k = 1$ universality class. Outside this region the system is presumably dimerized (or else goes into a ferromagnetic phase or possibly some more exotic phase). The multicritical points are likely to occur on the boundaries of the $k = 1$ region. The simplest case is $s = 3/2$. There is only one relevant operator $(\mathrm{tr}\, g)^2$ and the marginal operator $\mathbf{J}_L \cdot \mathbf{J}_R$. Thus we can consider a two dimensional parameter space. The two parameters could be, for example second nearest neighbor and biquadratic couplings. The anticipated phase diagram is shown in Figure 3.

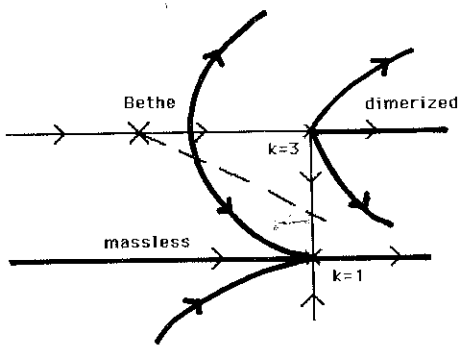


Fig. 3. The phase diagram for translation invariant $s = 3/2$ chains or $k = 3$ WZW models with $g \to -g$ symmetry. The arrows denote the flow of the coupling constants with increasing length scale. The dotted line is the type of path probably followed by the Hamiltonians in Fig. 4.

One boundary of the massless phase is determined by the vanishing of the marginal coupling. Thus the mass should grow exponentially slowly upon crossing this boundary. It contains the flow line from the $k = 3$ to the $k = 1$ fixed points. The other boundary is determined by the relevant coupling vanishing. Thus the mass should have power-law growth upon crossing it. Points along this boundary are attracted to the $k = 3$ fixed point. The integrable model presumably lies along such a boundary. In Fig. 4 we show a calculation [34] 1 of the conformal anomaly parameter, c, and $\eta \equiv 2x$, where x is the scaling dimension of $(-1)^i \mathbf{S}_i$, i.e. the WZW field, g. The leading logarithmic corrections have been subtracted off, in the case of η by measuring both singlet and triplet gaps. The calculations were done (on a Cyber) for $s = 3/2$ chains of $L \leq 12$ and a one-parameter family of Hamiltonians which included the integrable and Heisenberg Hamiltonians, as special cases. We see that $c = 9/5$ and $x = 3/10$, corresponding to $k = 3$, at the integrable point. Moving away from this point, we see crossover to $c = 1$ and $x = 1/2$, corresponding to $k = 1$. The Hamiltonian seems to cross a phase boundary at $\mu \approx 1.5$. This set of Hamiltonians presumably corresponds to a path like the one shown in Fig. 3.

Integer-s chains should have a similar looking phase diagram except that the massless $k = 1$ region is replaced by a massive region with unbroken symmetry. Outside this region exist dimerized phases (or ferromagnetic, etc.). The multicritical points presumably lie on the boundaries of the massive singlet phase. This seems to be consistent with numerical work on $s = 1$ chains in the vicinity of the integrable point [35], and variational [4] and large-N arguments [36].

It seems likely that, for integer or half-integer-s, $k = 2s$ is not the *only* multi-critical point that can occur; other (lower?) values of $k > 1$ should also be possible.

Note that an *explicit* breaking of the discrete symmetry (staggered interactions in the spin chain) will shift the minimum of the potential away from $|\varphi| = \pi$ and thus shift the topological angle away from π (or zero). Repeating the above calculation, we find a topological angle $\theta = k(|\varphi| - \sin|\varphi|)$. Thus a symmetry-breaking operator like $\mathrm{tr}(g)$ gains a topological significance in the low-energy theory. We expect shifting θ to produce a mass. It should lead to a $\mathrm{tr}(g)$ term in the $k = 1$ WZW model, of dimension $1/2$ and thus produce a gap $\propto |\theta - \pi|^{2/3}$ (up to logarithmic corrections). The observation that $\theta - \pi$ is a relevant coupling, and the $O(3)$ σ-model coupling constant, g, is marginal at the $k = 1$ fixed point implies a renormalization group flow diagram

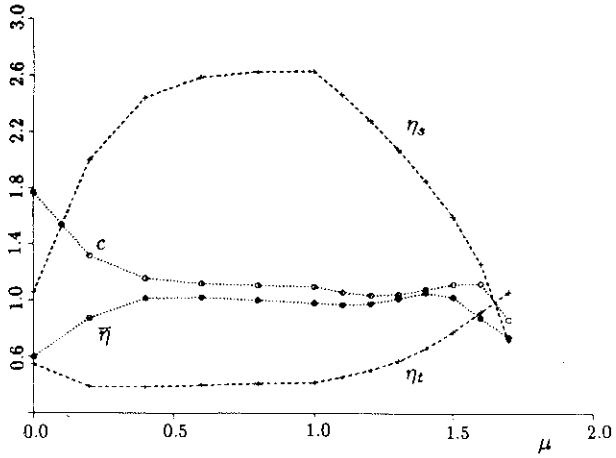


Fig. 4. Correlation exponent $\eta \equiv 2x$, measured from the triplet gap (η_t), singlet gap (η_s) and with leading logarithmic correction removed ($\bar{\eta}$) and c for a set of $s = 3/2$ Hamiltonians $H = (1-\mu)H_b + \mu H_1 (0 < \mu < 1)$ and $H = H_1 + [(\mu - 1)/2]H_2 (1 < \mu < 1.7)$. Here H_1 and H_2 are the nearest and second nearest neighbor Heisenberg Hamiltonians and H_b is the Bethe ansatz integrable Hamiltonian. $\eta = 3/5$ and 1 and $c = 9/5$ and 1 for the $k = 3$ and $k = 1$ WZW models respectively.

qualitatively like the one in Fig. 5.

Note that the lines $\theta = 0$ and $\theta = \pi$ are invariant under renormalization due to the $\varphi \to -\varphi$ symmetry at these values of θ. The renormalization at $g \to 0$ is independent of θ and is described by normal perturbation theory. All trajectories flow to non-universal, short-range fixed points except for the π trajectory which flows to the non-trivial critical point at a value of g_c of $O(1)$. The critical point separates phases of unbroken and spontaneously broken Z_2 symmetry. Off the lines $\theta = \pi$ or $\theta = 0$, Z_2 is explicitly broken so there is no phase transition.

The present analysis complements several other approaches to quantum spin chains. The argument that at large-s the low-energy spectrum of a quantum spin chain is described by the $O(3)$ σ-model with $\theta = 2\pi s$ is now seen to be true in a finite region in the space of spin Hamiltonians.

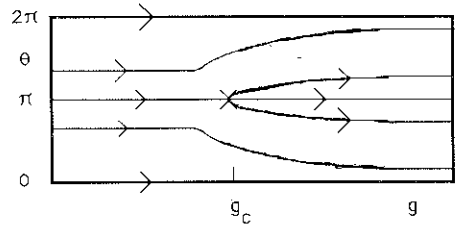


Fig. 5. Proposed renormalization group flow diagram for the $O(3)$ nonlinear σ-model. The arrows denote the flow of the coupling constants with increasing length scale.

It fails at the multicritical points and in the dimerized regions. Moving even infinitesimally away from one of the multicritical points into the undimerized region, it is again true. The, in many ways unfortunate, fact that the integrable models are examples of such multicritical points no longer appears to be a counter-example to the field theory arguments.

Furthermore, the conclusion that the critical theory for the $O(3)$ σ-model at $\theta = \pi$ is the $k = 1$ WZW model is consistent with Zamalodchikov's "c-theorem" [37]. This theorem states that if a field theory flows between two conformally invariant fixed points then the value of the conformal anomaly parameter c must be smaller at the infra-red stable fixed point. In this case the flow is from the unstable zero-coupling fixed point of the σ-model to the $k = 1$ fixed point. The unstable fixed point is a theory of two free (Goldstone) bosons with $c = 2$; the stable fixed point has $c = 1$.

There are not many known mechanisms in field theory (or in condensed matter physics) that will guarantee the stability of massless excitations in the presence of interactions. In four dimensions these include gauge invariance (which leads to a massless photon), spontaneous symmetry breaking (which leads to massless Goldstone bosons) and chiral symmetry (which if unbroken can, in some cases lead to massless fermions by the t'Hooft anomaly conditions). In two dimensions a massless phase can arise as a result of abelian or non-abelian chiral symmetry. In some field theories this symmetry is exact. In other cases (including all the condensed matter systems) it is not an exact symmetry of the mi-

croscopic model but is an effective symmetry of the critical theory, being broken only by irrelevant operators. The above arguments suggest that a topological mechanism plays a role in determining which Hamiltonians are attracted to the chirally invariant fixed point.

Appendix 1. Notation, free fermion and free boson field theories...

Notations:

(i) Free fermions:

$$\psi_{\mathrm{R}} = \int_0^\infty \frac{\mathrm{d}k}{\sqrt{2\pi}} \left(\mathrm{e}^{ik(t-x)} a_k + \mathrm{e}^{-ik(t-x)} b_k^\dagger \right),$$

$$\psi_{\mathrm{L}} = \int_{-\infty}^0 \frac{\mathrm{d}k}{\sqrt{2\pi}} \left(\mathrm{e}^{-ik(t+x)} a_k + \mathrm{e}^{+ik(t+x)} b_k^\dagger \right).$$

$$H = \psi_{\mathrm{R}}^\dagger i \frac{\mathrm{d}}{\mathrm{d}x} \psi_{\mathrm{R}} - \psi_{\mathrm{L}}^\dagger i \frac{\mathrm{d}}{\mathrm{d}x} \psi_{\mathrm{L}}, \quad \int \mathrm{d}x\, H = \sum_k |k| \left(a_k^\dagger a_k + b_k^\dagger b_k \right)$$

$$\frac{L = -i\psi^\dagger, \mathrm{d}}{\mathrm{d}t\psi - H. \quad \text{so } \frac{\delta L}{\delta \dot{\psi}} \equiv \Pi_\psi = -i\psi^\dagger, \quad \{\Pi_\psi, \psi\} = -i}$$

$$L = -i \left[\psi_R^\dagger \partial_+ \psi_R + \psi_{\mathrm{L}}^\dagger \partial_- \psi_{\mathrm{L}} \right]$$

$$x_\pm \equiv (x_0 \pm x_1)/2, \quad \partial_\pm \equiv \partial_0 \pm \partial_1 \quad (x_0 = t, \quad x_1 = -x^1 = -x)$$

$$\psi_{\mathrm{R}} = \psi_{\mathrm{R}}(x_+), \quad \psi_{\mathrm{L}} = \psi_{\mathrm{L}}(x_-).$$

$$\left\langle \psi_{\mathrm{R}}^\dagger(t,x) \psi_{\mathrm{R}}(0,0) \right\rangle = \int_0^\infty \frac{\mathrm{d}x}{2\pi} \mathrm{e}^{ik(t-x+i\varepsilon)} = \frac{i}{4\pi(x_+ + i\varepsilon)}$$

$$\left\langle \psi_{\mathrm{L}}^\dagger(t,x) \psi_{\mathrm{L}}(0,0) \right\rangle = \frac{i}{4\pi(x_- + i\varepsilon)}$$

$$J_{\mathrm{R}} \equiv : \psi_{\mathrm{R}}^\dagger \psi_{\mathrm{R}} :, \quad J_{\mathrm{L}} \equiv : \psi_{\mathrm{L}}^\dagger \psi_{\mathrm{L}} :$$

$$\langle J_{\mathrm{R}}(x_+) J_{\mathrm{R}}(0) \rangle = \left\langle < \psi_{\mathrm{R}}^\dagger(x_+) \psi_{\mathrm{R}}(0) \right\rangle \left\langle \psi_{\mathrm{R}}(x_+) \psi_{\mathrm{R}}^\dagger(0) \right\rangle$$

$$= \left(\frac{i}{4\pi x_+} \right)^2 = -\frac{1}{(4\pi x_+)^2}$$

$$\left\langle \psi_{\mathrm{L}}^\dagger(\mathbf{x}) \psi_{\mathrm{R}}(\mathbf{x}) \psi_{\mathrm{R}}^\dagger(\mathbf{0}) \psi_{\mathrm{L}}(0) \right\rangle = \left(\frac{i}{4\pi x_+} \right) \left(\frac{i}{4\pi x_-} \right) = \frac{1}{4\pi^2 (x^2 - t^2)}$$

(positive for space-like separation).

Energy-momentum tensor:

$$J_{\mathrm{R}}(x_+) J_{\mathrm{R}}(x_+ + \varepsilon_+) =: \psi_{\mathrm{R}}^\dagger \psi_{\mathrm{R}} \psi_{\mathrm{R}}^\dagger \psi_{\mathrm{R}} :$$
$$+ \left[\psi_{\mathrm{R}}^\dagger(x_+) \psi_{\mathrm{R}}(x_+ + \varepsilon_+) - \psi_{\mathrm{R}}^\dagger(x_+ + \varepsilon_-) \psi_{\mathrm{R}}(x_+) \right]$$
$$< \psi_{\mathrm{R}}(x_+) \psi_{\mathrm{R}}^\dagger(x_- + \varepsilon_-) >$$
$$+ \text{constant} \to -\frac{i}{4\pi} \left[\psi_{\mathrm{R}}^\dagger \partial_- \psi_{\mathrm{R}} - \partial_- \psi_{\mathrm{R}}^\dagger \psi_{\mathrm{R}} \right].$$

Thus $T_{\mathrm{R}} = \pi \lim_{\varepsilon \to 0} [J_{\mathrm{R}}(x_+) J_{\mathrm{R}}(x_+ + \varepsilon_+) + \text{constant}]$, where

$$T_{\mathrm{L,R}} \equiv (H \pm P)/2,$$

H and P being the Hamiltonian and momentum operators. Similarly:

$$T_{\mathrm{L}} \approx \pi J_{\mathrm{L}} J_{\mathrm{L}}.$$

SU(2) case:

$$J_{\mathrm{L,R}} \equiv : \psi_{\mathrm{L,R}}^{+\alpha} \psi_{\mathrm{L,R}\alpha} :,$$

$$\mathbf{J}_{\mathrm{L,R}} \equiv \psi_{\mathrm{L,R}}^{+\alpha} \frac{1}{2} \boldsymbol{\sigma}_\alpha^\beta \psi_{\mathrm{L,R}\beta},$$

$$T_{\mathrm{L}} = \frac{\pi}{2} J_{\mathrm{L}} J_{\mathrm{L}} + \frac{2\pi}{3} \mathbf{J}_{\mathrm{L}} \cdot \mathbf{J}_{\mathrm{L}}$$

(ii) Bosons: $\varphi \equiv \varphi_{\mathrm{L}} + \varphi_{\mathrm{R}}$

$$\varphi_{\mathrm{R}} = \int_0^\infty \frac{\mathrm{d}k}{\sqrt{4\pi k}} \left[\mathrm{e}^{ik(t-x)} a_k + \mathrm{e}^{-ik(t-x)} a_k^\dagger \right]$$

$$\varphi_{\mathrm{L}} = \int_{-\infty}^0 \frac{\mathrm{d}k}{\sqrt{4\pi|k|}} \left[\mathrm{e}^{-ik(t+x)} a_k + \mathrm{e}^{ik(t+x)} a_k^\dagger \right]$$

$$\partial_- \varphi_{\mathrm{R}}(x_+) = 2i \int_0^\infty \frac{\mathrm{d}k}{\sqrt{4\pi k}} k \left[\mathrm{e}^{ik(t-x)} a_k - \mathrm{e}^{-ik(t-x)} a_k^\dagger \right]$$

$$\langle \partial_- \varphi_{\mathrm{R}}(x_+) \partial_- \varphi_{\mathrm{R}}(0) \rangle = 4 \int \frac{\mathrm{d}k}{4\pi} k \mathrm{e}^{ik(t-x)}$$
$$= -4i \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \frac{\mathrm{d}k}{4\pi} \mathrm{e}^{ik(t-x)} = -4i \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) \left(\frac{-1}{4\pi i (t-x)} \right)$$
$$= -1/\pi (t-x)^2 = -\frac{1}{4\pi x_-^2}$$

$$\langle \partial_+ \varphi_L(x_-)\, \partial_+ \varphi(0)\rangle = -\frac{1}{4\pi x_-^2}$$

$$\left\langle e^{i\beta\varphi_R(x_+)} e^{-i\beta\varphi_R(0)}\right\rangle \propto x_+^{-\beta^2/4\pi}$$

$$J_R = (4\pi)^{-1/2}\partial_-\varphi_R, \quad J_L = -(4\pi)^{-1/2}\partial_+\varphi_L$$

$$\psi_R \propto \exp\left(i\sqrt{4\pi}\varphi_R\right), \quad \psi_L \propto \exp\left(-i\sqrt{4\pi}\varphi_L\right)$$

$$\text{with} \quad \varphi = \varphi_L + \varphi_R,$$

$$\langle \varphi_R(x_+)\varphi_R(0)\rangle = -\frac{1}{4\pi}\ln ix_+,$$

$$\langle \varphi_L(x_+)\varphi_L(0)\rangle = -\frac{1}{4\pi}\ln ix_-$$

so:

$$\left\langle \exp\left[-i\sqrt{4\pi}\varphi_R(x_+)\right]\exp\left[i\sqrt{4\pi}\varphi_R(0)\right]\right\rangle \propto \exp\left[-\ln(-ix_+)\right] = \frac{i}{x_+}$$

Note that:

$$\lim_{\varepsilon\to 0} : \psi_R^\dagger(\varepsilon)\psi_R(0) := < \psi_R^\dagger(\varepsilon)\psi_R(0) > i\sqrt{4\pi}[\varphi_R(0) - \varphi_R(\varepsilon)]$$

$$\Rightarrow [\frac{i}{4\pi\varepsilon}][-i\sqrt{4\pi}\varepsilon\partial_-\varphi_R] = (4\pi)^{-1/2}\partial_R, \quad \text{etc.}$$

Anti-commutation relations:

$$\text{Use} \quad \frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial x} = \partial_-\varphi = \partial_-\varphi_R = -\frac{2\partial\varphi_R}{\partial x},$$

$$\varphi_R(x) = \frac{1}{2}\left[\varphi(x) - \int_{-\infty}^{x}\frac{\partial\varphi}{\partial t}\right]$$

$$\psi_R \propto \exp\left[i\sqrt{\pi}\varphi(x)\right]\exp\left[-i\sqrt{\pi}\int_{-\infty}^{x}\frac{\partial\varphi}{\partial t}\right].$$

Using $e^A e^B = e^B e^A e^{[A,B]}$, for $[A,B]$ a c-number, we have:

$$\psi_R(x)\psi_R(y) = \exp\left[i\sqrt{\pi}\varphi(x)\right]\exp\left[-i\sqrt{\pi}\int_{-\infty}^{x}\frac{\partial\varphi}{\partial t}\right]$$

$$\cdot \exp\left[i\sqrt{\pi}\varphi(y)\right]\exp\left[-i\sqrt{\pi}\int_{-\infty}^{y}\frac{\partial\varphi}{\partial t}\right]$$

$$= \exp\left[i\sqrt{\pi}\varphi(x)\right]\exp\left[i\sqrt{\pi}\varphi(y)\right]\exp\left[-i\sqrt{\pi}\int_{-\infty}^{x}\frac{\partial\varphi}{\partial t}\right]$$

$$\cdot \exp\left[-i\sqrt{\pi}\int_{-\infty}^{y}\frac{\partial\varphi}{\partial t}\right]\exp\left[i\pi\theta(x-y)\right].$$

Since this is an odd function of $x - y$, the anticommutator vanishes. Similarly for $\{\psi_R^\dagger(x),\ \psi_R(y)\}$, except that now there is a singular term from bringing the 2 fields to the same point. This contributes the needed $\delta(x-y)$.

Green's functions: 4-point function:
abbreviate $\psi_R(x_+)$ by $\psi(x)$:

$$\langle \psi^\dagger(x_1)\psi^\dagger(x_2)\psi(y_1)\psi(y_2)\rangle$$

$$= \frac{1}{(4\pi)^2}\left[\frac{1}{(x_1-y_1)(x_2-y_2)} - \frac{1}{(x_1-y_2)(x_2-y_1)}\right]$$

$$= \left(\frac{1}{(4\pi)^2}\right)\frac{(x_1-x_2)(y_1-y_2)}{(x_1-y_1)(x_2-y_2)(x_1-y_2)(x_2-y_1)}.$$

On the other hand, setting $\varphi_R(x_+)$ to $\varphi(x)$

$$< \exp\left[-i\sqrt{4\pi}\varphi(x_1)\right]\exp\left[-i\sqrt{4\pi}\varphi(x_2)\right]$$

$$\exp\left[i\sqrt{4\pi}\varphi(y_1)\right]\exp\left[i\sqrt{4\pi}\varphi(y_2)\right] >$$

$$\propto \exp\{2\pi[G(x_1-y_1) + G(x_1-y_2) + G(x_2-y_1)$$

$$+G(x_2-y_2) - G(x_1-x_2) - G(y_1-y_2)]\},$$

where $G(x)$ is the Green's function $(-1/2\pi)\ln ix$. These are the same. In fact this can be proved for an arbitrary n-point function.

Energy-momentum tensor:

$$T_R = \frac{1}{4}(\partial_-\varphi_R)^2 = \pi J_R J_R.$$

SU(2) case:

$$S_{WZW}(g) = \frac{1}{8\pi}\int d^2x\, \mathrm{tr}\,\partial_\mu g^\dagger \partial^\mu g$$

$$+ \frac{1}{12\pi}\int d^3x \varepsilon^{\mu\nu\lambda}\,\mathrm{tr}\, g^\dagger\partial_\mu g g^\dagger \partial_\nu g g^\dagger \partial_\lambda g.$$

The second (Wess–Zumino) term is defined by extending the two-dimensional space (or space-time) to a three-dimensional half-space ($x_3 < 0$) and $g(x_\mu)$ is an arbitrary extrapolation of the function defined on the two-dimensional space (at $x_3 = 0$) which approaches **1** at

$x_3 \to -\infty$. The boundary value determines the Wess–Zumino term up to a term of the form $2\pi n$ (or $2\pi i n$ in Euclidean Space) where n is an integer. Thus the path-integral is well defined if the coefficient in front of this term is an integer. (Here we have the lowest possible integer, $k = 1$.) The coefficient of the first term is then fixed by the condition of chiral symmetry or equivalently by the fixed point of the renormalization group.

Chiral symmetry:

$$g \to UgV.$$

The currents are written:

$$J_{\rm L} = -\frac{1}{\sqrt{4\pi}}\partial_+\varphi, \qquad J_{\rm R} = \frac{1}{\sqrt{4\pi}}\partial_-\varphi$$
$$\mathbf{J}_{\rm L} = -\frac{i}{4\pi}\,\mathrm{tr}\,\partial_+ g g^\dagger \sigma, \quad \mathbf{J}_{\rm R} = \frac{i}{4\pi}\,\mathrm{tr}\, g^\dagger \partial_- g \boldsymbol{\sigma}$$

and the left-right fermion product as

$$\psi_{\rm L}^{+\alpha}\psi_{R\beta} \propto g^\alpha_\beta \exp\left(i\sqrt{2\pi}\varphi\right).$$

The energy-momentum tensor of the WZW model takes the Sugarawa form:

$$T_{\rm R} = \frac{2\pi}{3}\mathbf{J}_{\rm R}\cdot\mathbf{J}_{\rm R}$$

g has dimensions 1/2:

$$\left\langle g^\alpha_\beta(\mathbf{x}) g^{+\delta}_\gamma(\mathbf{y})\right\rangle \propto \delta^\alpha_\gamma\delta^\delta_\beta/|\mathbf{x}-\mathbf{y}|.$$

Thus g and $\exp(i\sqrt{2\pi}\varphi)$ each carry 1/2 the total scaling dimension, 1 of the fermion bilinear.

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COURSE 11

STRINGS...

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5630 South Ellis Ave., Chicago, IL 60637 USA

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