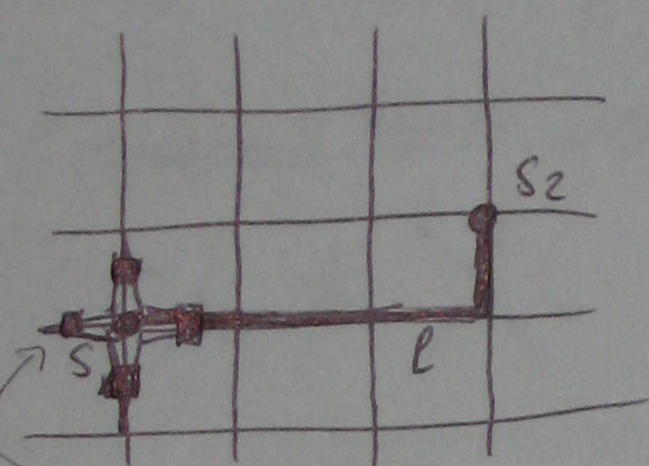
Lecture 2

FAQ regarding the previous lecture

Q. Why does the path operator create a pair of charges?

A.



$$W_\ell = \prod_{j \in \ell} \sigma_j^z$$

$$|\Psi_{s_1 s_2}\rangle = W_\ell |\Psi_0\rangle$$

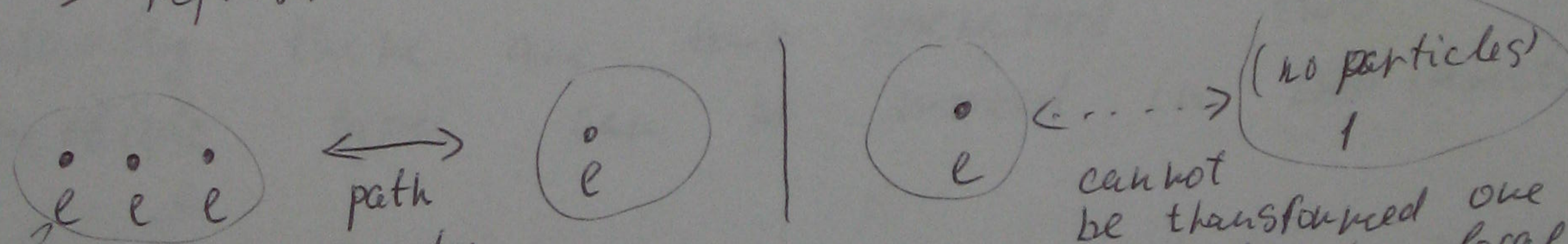
$$A_{s_1} = \prod_{j \in \mathrm{star}(s_1)} \sigma_j^x$$

$$W_\ell A_{s_1} = -A_{s_1} W_\ell$$

$$\boxed{\begin{aligned} A_{s_1} |\Psi_{s_1 s_2}\rangle &= A_{s_1} W_\ell |\Psi_0\rangle = \\ &= -W_\ell A_{s_1} |\Psi_0\rangle = -W_\ell |\Psi_0\rangle = \\ &= -|\Psi_{s_1 s_2}\rangle \end{aligned}}$$

Q. What are these labels: $1, e, m, \varepsilon$?

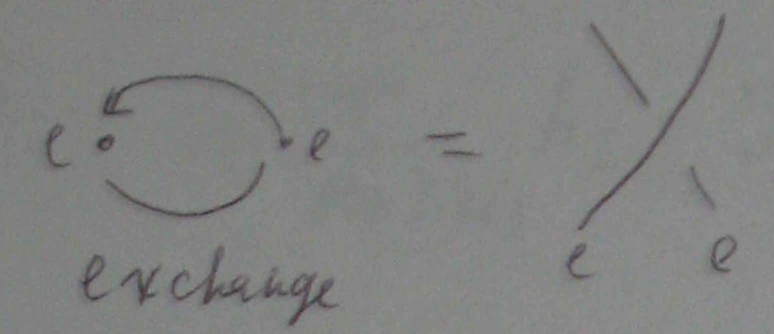
A. These are superselection sectors = equivalence classes of excited states w.r.t. local operators = representations of the operator algebra.

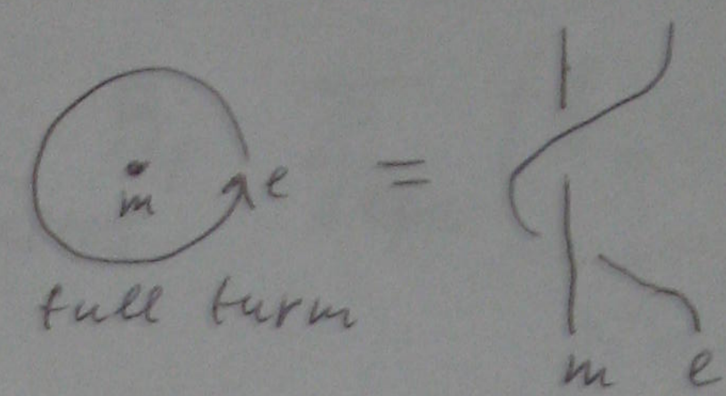


this is a local excited state.

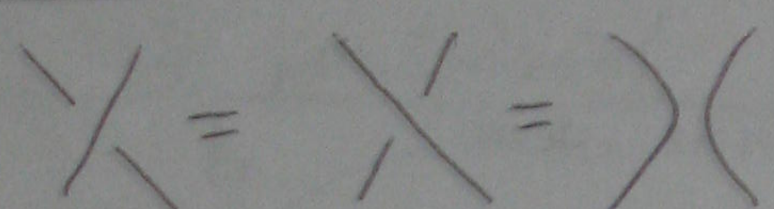
Q. How does one read a braid diagram?

A.

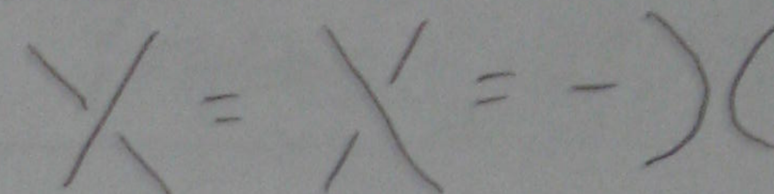




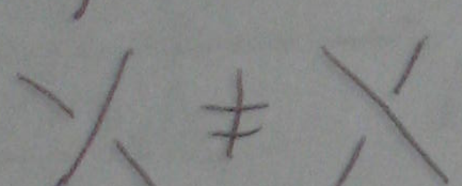
<u>bosons</u>



<u>fermions</u>

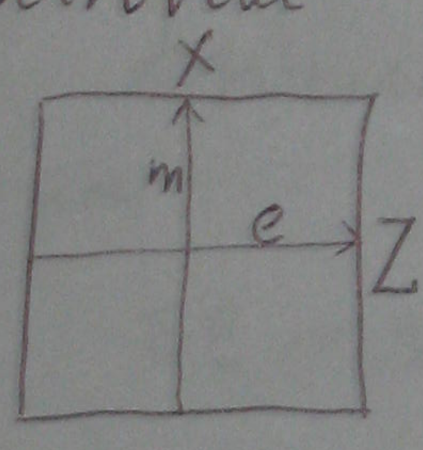


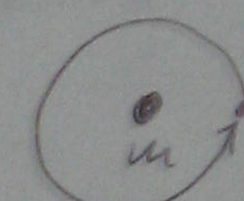
<u>anyons with nontrivial self-statistics</u> (not present in our model)



<u>Einarsson's argument</u> (1990)

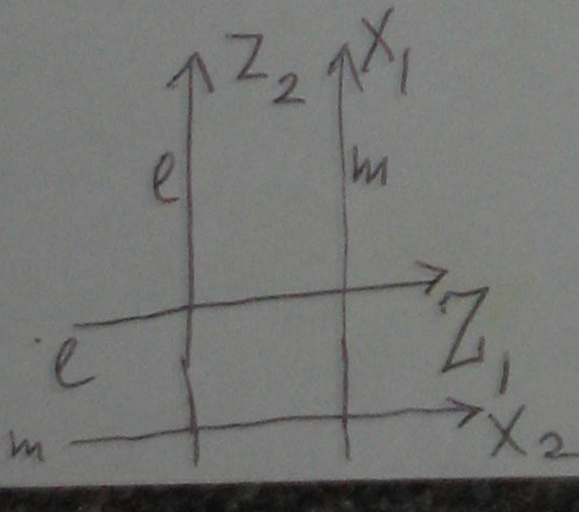
The degeneracy on the torus <u>follows</u> from nontrivial statistics



$Z^{-1}X^{-1}ZX =$  $= -1$

These are two noncommuting operators acting on the ground space $\mathcal{L} \Rightarrow$ $\dim \mathcal{L} > 1$

Actually, there are four operators: two particle types can be moved in two directions.



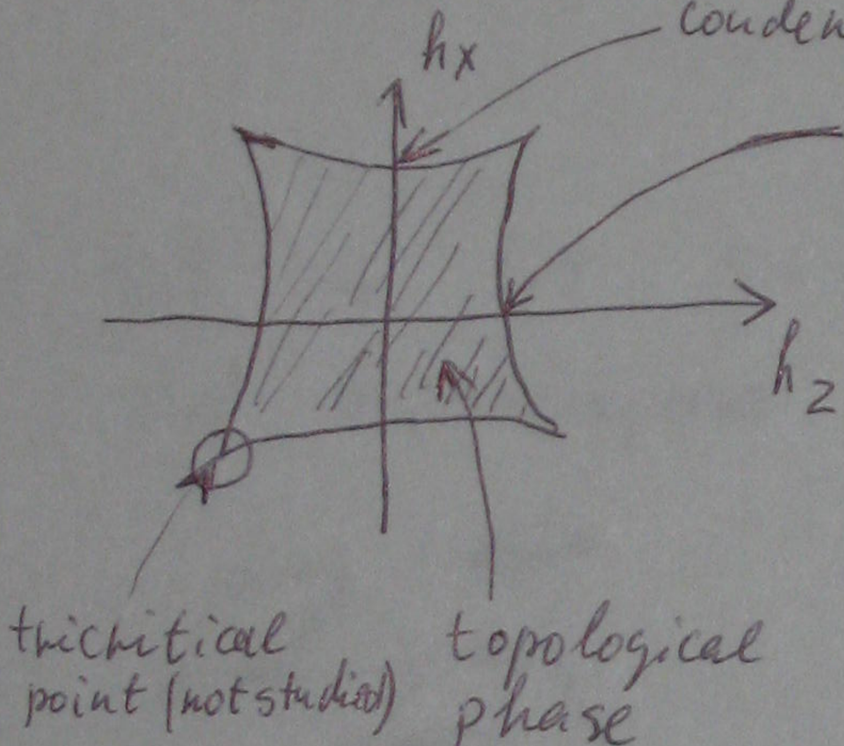
The commutation relations imply that $\dim \mathcal{L} = 4$

Perturbation analysis

$$H = -J_e \sum_{\text{vertices}} A_s - J_m \sum_{\text{plaquettes}} B_p - \sum_{\text{edges}} \underbrace{\left(h_x \sigma_j^x + h_z \sigma_j^z \right)}_{\text{perturbation}}$$

$h_z \neq 0 \Rightarrow$ electric charges can hop from site to site

$\Rightarrow$ nontrivial dispersion $\left(\varepsilon(q) \approx 2J_e - 2h_z (\cos q_x + \cos q_y) \right)$



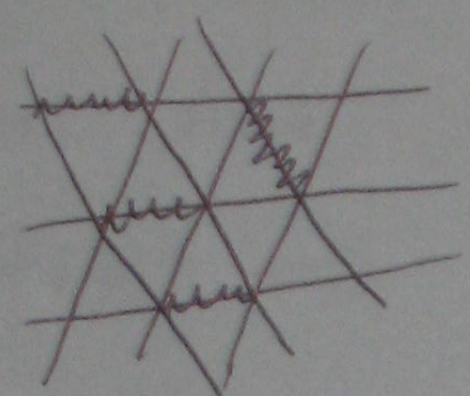
Bose-condensation of charges
vortices become confined
(Equivalent to the transition in a 3-D Ising model)

Within the topological phase, the ground state degeneracy is preserved with exponential precision:

$$\delta E \sim \Delta \underbrace{\exp(-L/\xi)}_{\text{due to virtual quasiparticle tunneling:}}$$

the only way to act upon the ground state is to wind a quasiparticle around the torus

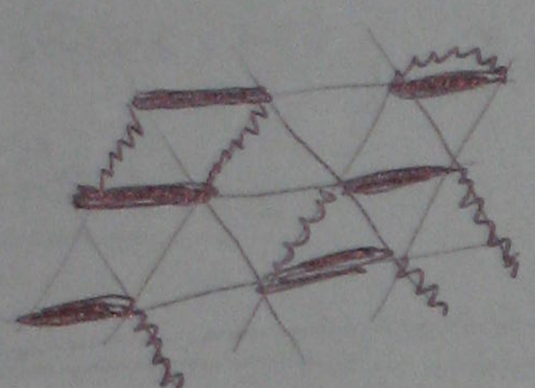
Analogy to the dimer model on the triangular lattice



Dimer liquid =

= uniform superposition of all $\underbrace{\text{dimer covers}}_{\text{all vertices are covered}}$

Let us fix some reference configuration and superimpose an arbitrary configuration



$x_j = 0$ or

$x_j = 1$ or

$\boxed{u_s(x) = \sum_{j \in \text{star}(s)} x_s = 0 \pmod 2}$

$$|\Psi_{\text{dimer}}\rangle = \sum_{\substack{\text{some loop} \\ \text{conf.}}} |\vec{x}\rangle$$

Now, consider

$$|\Psi_0\rangle = \sum_{\substack{\vec{x}:\, u_s(\vec{x}) = 0 \\ \text{for all } s}} |\vec{x}\rangle$$

This is the toric code state in the dual basis

$x = 0 \quad = \quad \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$

$u_s(\vec{x}) = 0 \iff A_s|\Psi\rangle = |\Psi\rangle$

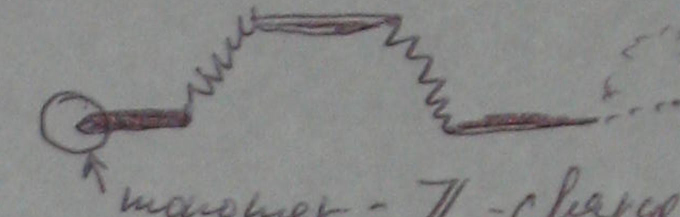
$x = 1 \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

[illegible] = Broken loop [illegible]

Properties of the dimer state are similar, but not identical, to the toric code state.

Not so on the square lattice:

Broken loop = Dirac string for a monomer



Loops have <u>direction</u> $\Rightarrow$ integer charges $\Rightarrow$ $\underbrace{\mathbb{Z}_2 \to U(1)}$,

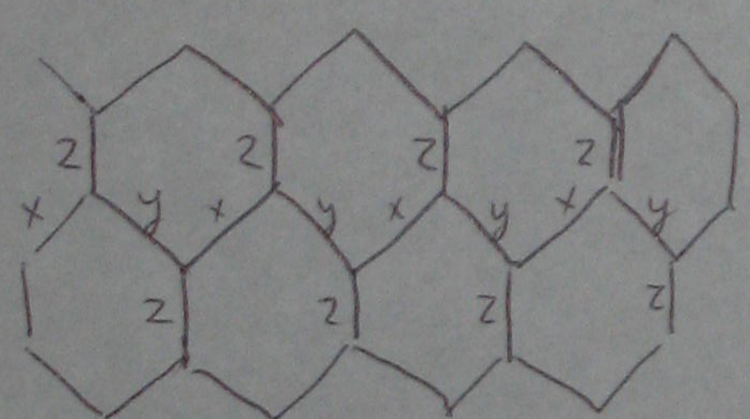
It is known that the $U(1)$-theory (= electrodynamics) is unstable in 2D.

Vortices have arbitrary value in $U(1)$ $\Rightarrow$ no gap.

Moreover, the vortices spontaneously condense.

<u>Reason</u>: $S_{monopole} < \infty$ (Polykov)

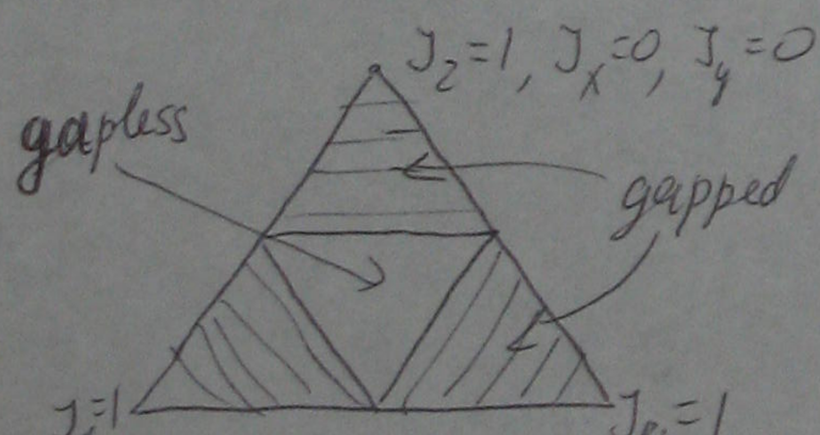
The honeycomb lattice model



<u>Spins on the sites</u>

$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z$$

<u>Answer</u>: The signs of J_x, J_y, J_z don't matter



The gapped phases are in the same univ. class as the toric code.

<u>But</u>: subtle breaking of the translational symmetry.