Lecture 1

Introduction

I will speak about topological properties of strongly correlated systems. I will focus on 2-dimensional systems with an energy gap.

- 2D – because certain phenomena (namely, nontrivial statistics) only occur it two dimensions
- Energy gap – because such systems are simpler:
  - only algebra and topology;
  - $\langle \sigma_j \sigma_k \rangle \sim \exp(-r_{jk}/\xi)$
  - quasiparticle statistics is well-defined.

Where do topological phases occur?

Fractional quantum Hall effect (FQHE)

$\nu = \frac{1}{3}, \frac{2}{5}$, etc. – Abelian phases (certain)

$\nu = \frac{5}{2}$ – non-Abelian phase (very likely)

Spin liquids:

Idea by Anderson (1973)

The quest is still open.

We need to understand what we are looking for!

Strategy:

1) Exactly solvable models.

2) Algebraic description of universality classes.

Lecture 1 "Toric code" ($\mathbb{Z}_2$-gauge model) and dimer models.

Lecture 2 Honeycomb lattice model

Lecture 3 Unpaired Majorana modes and non-Abelian anyons.

General properties of topological phases

- Unusual quasiparticles:
  - Fractional charge / spin or completely new quantum numbers

    Conservation laws (fusion rules) $\Longrightarrow$ "Symmetry" (not a group)

  - Unusual statistics

e.g. $\mathbb{Z}_2$-vortices (visons)
# of such vortices is conserved modulo 2, but there is no symmetry in the Hamiltonian

$$|\Psi\rangle \Rightarrow e^{i\varphi_{ab}}|\Psi\rangle \quad \text{- Abelian anyons}$$

$$|\Psi\rangle \rightarrow U_{ab}|\Psi\rangle \quad \text{- non-Abelian anyons}$$

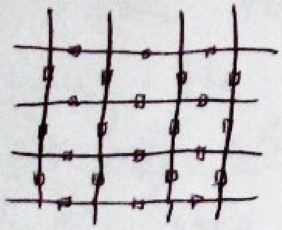
- Degenerate ground states (on the torus)

$$\delta E \sim \Delta \exp(-L/\xi)$$

The degeneracy is stable to local perturbations $\Longrightarrow$
$\Longrightarrow$ protected qubit!

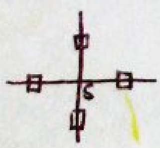
Degeneracy also occurs in a system of non-Abelian anyons on the plane.

"Toric code" ($\mathbb{Z}_2$-gauge model)



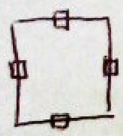
- Square lattice (on the plane – the torus will appear later)
- Spins on the edges (spin $\frac{1}{2}$)

$$\boxed{H = -J_e \sum_{\text{vertices}} A_s - J_m \sum_{\text{plaquettes}} B_p}$$

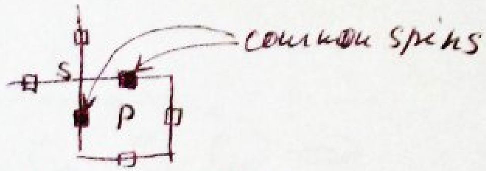


$$\boxed{A_s = \prod_{j \in \text{star}(s)} \sigma_j^x}$$

$$A_s B_p = B_p A_s$$



$$\boxed{B_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z}$$



$$\sigma_j^x \sigma_j^z = -\sigma_j^z \sigma_j^x$$

Two minus signs cancel

Let us describe basis states by variables

$$z_j = \begin{cases} 0 & \text{– spin up } \uparrow \\ 1 & \text{– spin down } \downarrow \end{cases}$$

~~(projections onto the~~ (relative to the z axis)

$$\vec{z} \overset{\text{def}}{=} (z_1, \ldots, z_N)$$

$$w_p(\vec{z}) = \sum_{j \in \text{boundary}(p)} z_j \pmod 2$$

vorticity ($\mathbb{Z}_2$ analogue of $\vec{\nabla} \times \vec{A}$)

Ground state:

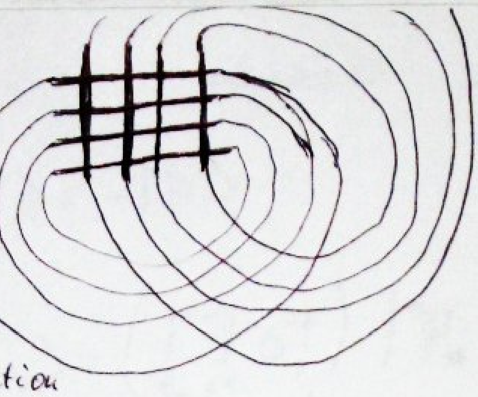
$$|\Psi_*\rangle = \sum_{\vec{z}:\ \vec{w}_p(z) = 0 \text{ for all } p} c_z |z\rangle$$

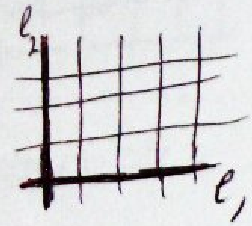
$$A_s|\Psi_*\rangle = |\Psi_*\rangle, \quad B_p|\Psi_*\rangle = |\Psi_*\rangle$$

"stabilizer conditions" (constraints)

A_s flips the spins $\Longrightarrow$ $c_z = \text{const}$

On the torus, the spin flips preserve the cohomology class of the spin configuration





$$W_l(\vec{z}) = \sum_{j \in l} z_j \pmod 2$$

Conserved numbers:

$$w_1 \equiv W_{l_1}(\vec{z}), \quad w_2 \equiv W_{l_2}(\vec{z})$$

Ground state:

$$|\Psi\rangle = \sum_{\substack{\vec{z}:\, \vec{W}_p(z)=0 \\ \text{for all } p}} C_{w_1, w_2} |z\rangle$$

Four independent parameters: $C_{00}, C_{01}, C_{10}, C_{11}$ $\Rightarrow$

$\Rightarrow$ four-dimensional ground space

("Physical" realization of a quantum error-correcting code) Kitaev 1997

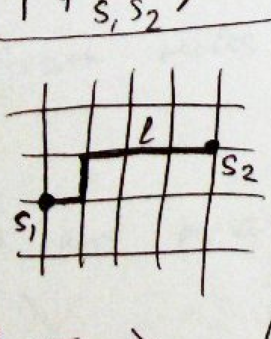
Excitations in the "toric code" model

Again, let us consider the model on the plane first.

$|\Psi_{s_1 s_2}\rangle$: $A_{s_1} |\Psi_{s_1 s_2}\rangle = -|\Psi_{s_1 s_2}\rangle$

two "electric charges" at sites s_1, s_2

$A_{s_2} |\Psi_{s_1, s_2}\rangle = -|\Psi_{s_1, s_2}\rangle$

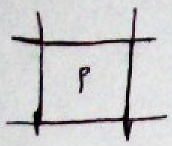


(The other constraints are satisfied with the + sign)

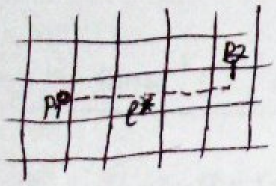
$$|\Psi_{s_1, s_2}\rangle = \Big(\prod_{j \in l} \sigma_j^z\Big) |\Psi_*\rangle$$

path operator,

Another excitation type: "magnetic charges" (vortices)

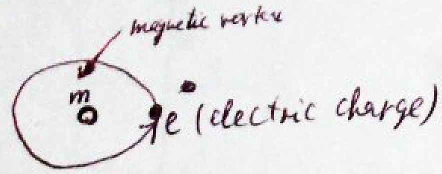


$$B_p|\Psi\rangle = -|\Psi\rangle$$

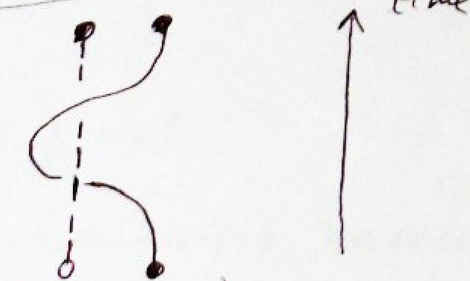


$$|\Psi_{p_1 p_2}\rangle = \underbrace{\left(\prod_{j\in \ell^*} \sigma_j^x\right)}_{\text{dual path operator}} |\Psi_*\rangle$$

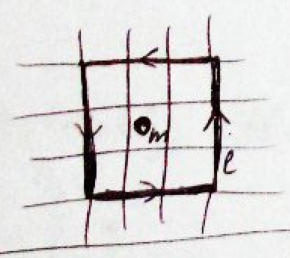
Quasiparticle statistics; superselection sectors and fusion rules



, or



(top view) (side view)



$$|\Psi\rangle \mapsto \left(\prod_{j\in\ell} \sigma_j^z\right)|\Psi\rangle = -|\Psi\rangle$$

$\boxed{|\Psi\rangle \to -|\Psi\rangle} \Longrightarrow$ e and m have nontrivial mutual statistics

Superselection sectors: $1, e, m, \underbrace{\varepsilon = e\times m}_{\text{dyon}}$

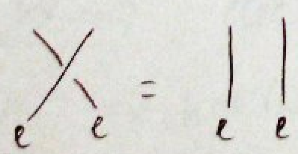
Fusion rules:

$e\times e = 1 \qquad e\times m = \varepsilon$

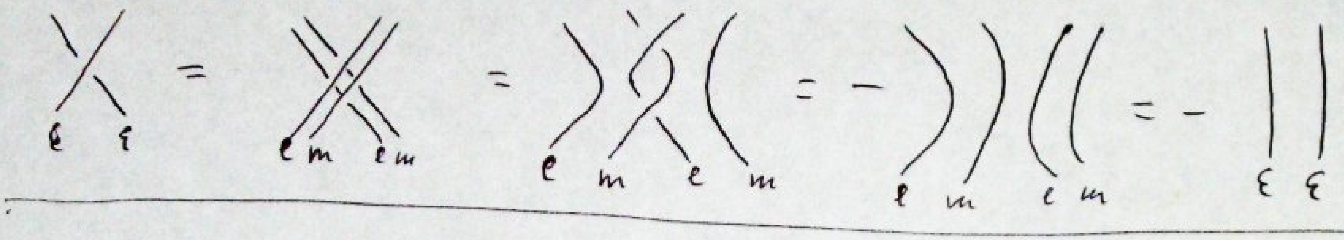
$m\times m = 1 \qquad e\times \varepsilon = m$

$\varepsilon\times\varepsilon = 1 \qquad m\times\varepsilon = e$

e and m are boson (if considered separately)

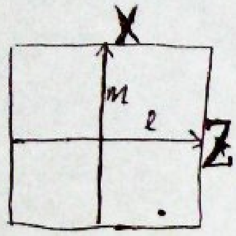


But ε's are fermions!



Einarsson's argument (1990)

The degeneracy on the torus follows from nontrivial statistics.



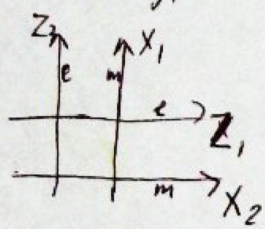
$Z^{-1}X^{-1}ZX =$ 

$= -1$

There are two noncommuting operators acting on the ground space $\Rightarrow$

$\dim \mathcal{L} > 1$

Actually, there are four operators (two particle types can be moved in two directions)



The commutation relations imply that $\dim \mathcal{L} = 4$

Perturbation analysis

$$H = -J_e \sum_{\text{vertices}} A_s - J_m \sum_{\text{plaquettes}} B_p - \underbrace{\sum_{\text{edges}} \left(h_x \sigma_i^x + h_z \sigma_{ij}^z\right)}_{\text{perturbations}}$$

$h_z \neq 0 \Rightarrow$ electric charges can hop from site to site
$\Rightarrow$ nontrivial dispersion $\left(\varepsilon(q) \approx 2J_e - 2h_z(\cos q_x + \cos q_y)\right)$