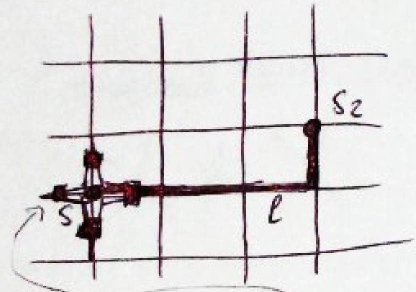
# Lecture 2

## FAQ regarding the previous lecture

Q. Why does the path operator create a pair of charges?

A.




$$W_e = \prod_{j \in e} \sigma_j^z$$

$$|\Psi_{s_1 s_2}\rangle = W_e |\Psi_0\rangle$$

$$A_{s_1} = \prod_{j \in \mathrm{star}(s_1)} \sigma_j^x$$

$$W_e A_{s_1} = -A_{s_1} W_e$$

$$\boxed{\begin{aligned} A_{s_1}|\Psi_{s_1 s_2}\rangle &= A_{s_1} W_e |\Psi_0\rangle = \\ &= -W_e A_{s_1}|\Psi_0\rangle = -W_e|\Psi_0\rangle = \\ &= -|\Psi_{s_1 s_2}\rangle \end{aligned}}$$

Q. What are these labels: $1, e, m, \varepsilon$?

A. These are superselection sectors = equivalence classes of excited states w.r.t. local operators = representations of the operator algebra.

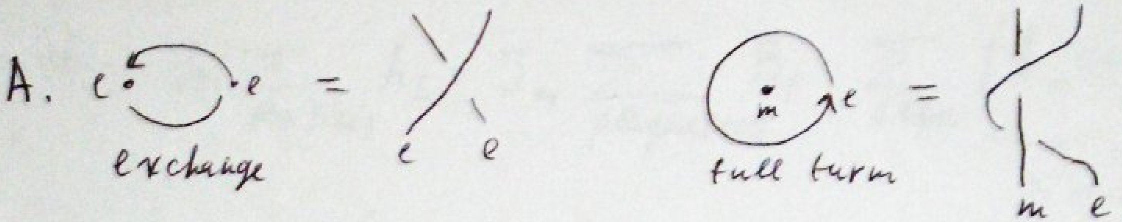
$e \; e \; e \xleftrightarrow{\text{path operator}} e$ | $e \xleftrightarrow{\cdots} 1$ (no particles)

this is a local excited state.

cannot be transformed one to another by local operators

Q. How does one read a braid diagram?

A.




bosons

fermions


anyons with nontrivial self-statistics (not present in our model)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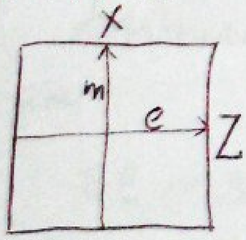
Einarsson's argument (1990)

The degeneracy on the torus follows from nontrivial statistics

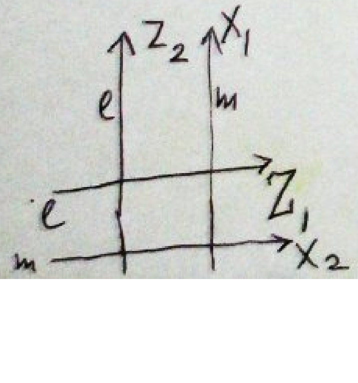



$Z^{-1}X^{-1}ZX =$  $= -1$



These are two noncommuting operators acting on the ground space $\mathcal{L} \Rightarrow$ $\dim \mathcal{L} > 1$

Actually, there are four operators: two particle types can be moved in two directions.



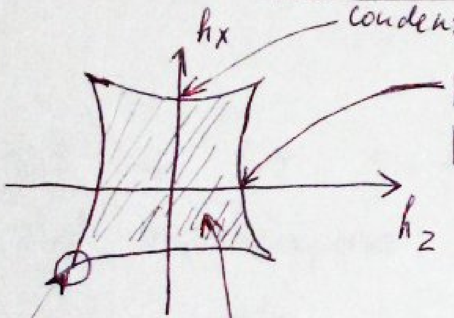

The commutation relations imply that $\dim \mathcal{L} = 4$

# Perturbation analysis

$$H = -J_e \sum_{\text{vertices}} A_s - J_m \sum_{\text{plaquettes}} B_p - \underbrace{\sum_{\text{edges}} \left( h_x \sigma_j^x + h_z \sigma_j^z \right)}_{\text{perturbation}}$$

$h_z \neq 0 \Rightarrow$ electric charges can hop from site to site

$\Rightarrow$ nontrivial dispersion $\left(\varepsilon(q) \approx 2J_e - 2h_z(\cos q_x + \cos q_y)\right)$




Bose-condensation of charges
vortices become confined
(Equivalent to the transition in a 3-D Ising model)

tricritical point (not studied)

topological phase

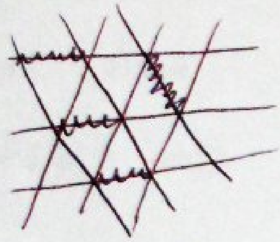
Within the topological phase, the ground state degeneracy is preserved with exponential precision:

$$\delta E \sim \Delta \underbrace{\exp(-L/\xi)}_{}$$

due to virtual quasiparticle tunneling:

the only way to act upon the ground state is to wind a quasiparticle around the torus

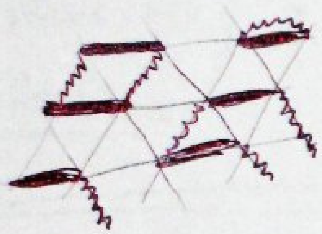
## Analogy to the dimer model on the triangular lattice



**Dimer liquid** =

= uniform superposition of all $\underbrace{\text{dimer covers}}_{\substack{\text{all vertices}\\ \text{are covered}}}$

Let us fix some reference configuration and superimpose an arbitrary configuration



$x_j = 0$ or

$x_j = 1$ or

$$\boxed{u_s(x) = \sum_{j \in \mathrm{star}(s)} x_s = 0 \pmod 2}$$

$$|\Psi_{dimer}\rangle = \sum_{\substack{\text{some loop}\\ \text{conf.}}} |\vec{x}\rangle$$

Now, consider

$$|\Psi_0\rangle = \sum_{\substack{\vec{x} : u_s(\vec{x}) = 0\\ \text{for all } s}} |\vec{x}\rangle$$

This is the toric code state in the dual basis

$$x = 0 \quad = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$$

$u_s(\vec{x}) = 0 \Longleftrightarrow A_s|\Psi\rangle = |\Psi\rangle$

$$x = 1 \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

~~[illegible]~~ loops = Broken loop

Properties of the dimer state are similar, but not identical, to the toric code state.

Not so on the square lattice!

Loops have <u>direction</u> $\Rightarrow$ integer charges $\Rightarrow$ $\underbrace{\mathbb{Z}_2 \to U(1)}$

Broken loop = Dirac string for a monomer

monomer = $\mathbb{Z}_2$-charge

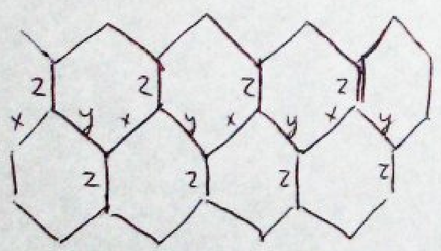
It is known that the $U(1)$-theory (= electrodynamics) is unstable in 2D.

Vortices have arbitrary value in $U(1)$ $\Rightarrow$ no gap.

Moreover, the vortices spontaneously condense.

<u>Reason</u> : $S_{monopole} < \infty$ (Polykov)

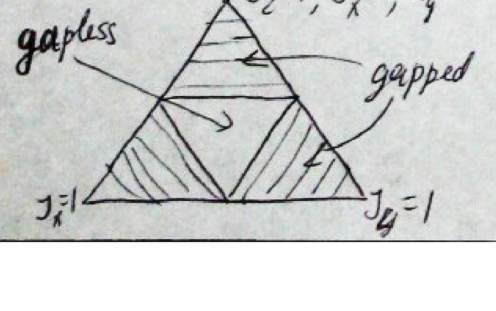
<u>The honeycomb lattice model</u>




<u>Spins on the sites</u>

$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z$$

<u>Answer</u> : The signs of $J_x, J_y, J_z$ don't matter




The gapped phases are in the same univ. class as the toric code.

<u>But</u> : subtle breaking of the translational symmetry.