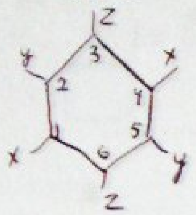
Lecture 3

How do we solve the problem?

Observation:



$$\underbrace{W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z}$$

commutes with every term in the Hamiltonian $\Longrightarrow$

an extensive number of integrals of motion.

W_p has eigenvalues ± 1 $\quad$ $W_p = 1$ – no vortex

$W_p = -1$ – vortex on hexagon p

Counting degrees of freedom

1 hexagon — 2 sites — 3 edges

m hexagons $\Longrightarrow$ $2m$ spins

$$\mathcal{H} = \bigoplus_{w_1,\dots w_m} \mathcal{L}_{w_1,\dots,w_m}$$

$\mathcal{L}_{w_1,\dots,w_m}$ ← common eigenspace of $W_1, \dots W_m$

= the space of states for the vortex configuration $(w_1, \dots w_m)$

$$\left.\begin{array}{l} \dim \mathcal{H} = 2^{2m} \\ 2^m \text{ vortex conf.} \end{array}\right\} \Longrightarrow \dim \mathcal{L}_{w_1,\dots,w_m} = \frac{2^{2m}}{2^m} = 2^m = (\sqrt{2})^{2m}$$

Fixing the vortex configuration is not the whole story. We are left with half a spin per site

<u>Guess</u>: These "half-spins" are <u>Majorana modes</u>

Indeed, the problem is solved by a reduction to free fermions

Majorana modes (operators)

Ordinary fermionic modes

$|0100\rangle$

$\underbrace{\qquad\qquad}_{m \text{ boxes}}$

each box can be empty or occupied by a fermion

$$a_k a_l + a_l a_k = 0$$
$$a_k a_l^+ + a_l^+ a_k = \delta_{kl}$$

Let us introduce self-adjoint operators

$$\boxed{\begin{aligned} c_{2k-1} &= a_k + a_k^+ \\ c_{2k} &= \frac{a_k - a_k^+}{i} \end{aligned}}$$

$$\boxed{\begin{aligned} &c_j^+ = c_j \\ &c_j^2 = 1 \\ &c_j c_l = -c_l c_j \text{ if } j \neq l \end{aligned}}$$

All $2m$ operators are treated on equal footing

$$\underbrace{c_j c_l + c_l c_j = 2\delta_{jl}}_{\text{Clifford algebra relation}}$$

Redundant representation of a spin

c, b^x, b^y, b^z are Majorana modes

Equivalent to 2 full fermionic modes $\Rightarrow$

$\Rightarrow$ $2^2 = 4$ states. $\square\ \square$

Let us impose a constraint

$$\boxed{D|\psi\rangle = |\psi\rangle}, \quad \text{where } \boxed{D = b^x b^y b^z c}$$

constrained (physical) space $\mathcal{M} \subset \tilde{\mathcal{M}}$ ← unconstrained (extended) space

$$\boxed{\begin{aligned} \tilde\sigma^x &= i b^x c \\ \tilde\sigma^y &= i b^y c \\ \tilde\sigma^z &= i b^z c \end{aligned}}$$

(1) $[\tilde\sigma^\alpha, D] = 0 \iff \tilde\sigma^\alpha$ preserve the physical subspace

(2) $(\tilde\sigma^\alpha)^2 = 1$,

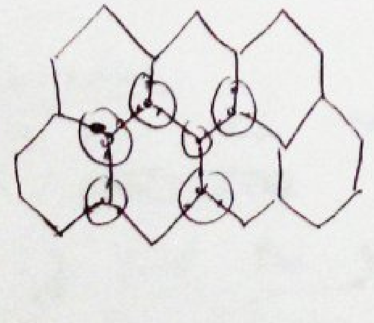
$$\tilde\sigma^x \tilde\sigma^y \tilde\sigma^z = i \underbrace{b^x b^y b^z c}_{D} \equiv i$$

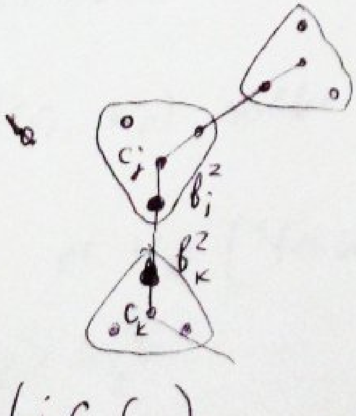
equal on the physical subspace

The reduction of the original model

Any spin Hamiltonian $H\{\sigma_j^\alpha\}$ can be represented by $\tilde H\{b_j^\alpha, c^\alpha\}$ subjected to the constraints $D_j|\Psi\rangle = |\Psi\rangle$.

For our Hamiltonian, this representation is actually useful.





$$\sigma_j^\alpha \sigma_k^\alpha \longrightarrow \tilde\sigma_j^\alpha \tilde\sigma_k^\alpha = \underbrace{(i b_j^\alpha b_k^\alpha)} (i c_j c_k)$$

All such operators commute because each b_j^α belongs to a single link.

$$\tilde{H} = \frac{i}{4}\sum_{jk}\hat{A}_{jk}\, c_j c_k$$

$$\hat{A}_{jk} = \begin{cases} 2J_{\alpha_{jk}}\hat{u}_{jk} & \text{if } j \text{ and } k \text{ are connected} \\ 0 & \text{otherwise}\end{cases}$$

$\hat{u}_{jk} = i\, b_j^{\alpha_{jk}} b_k^{\alpha_{jk}}$ ← link direction

Key trick We may replace $\hat{u}_{jk}$ by numbers

$u_{jk} = \pm 1$

$\prod_{p} u_{jk} = W_p$ - vorticity on plaquette p.

$\boxed{u_{jk} \text{ is a } \mathbb{Z}_2\text{-vector potential}}$

Thus, $\boxed{H_u = \frac{i}{4}\sum_{jk} A_{jk} c_j c_k}$ - Quadratic fermionic Hamiltonian

Procedure

1) For each vortex conf. $(w_1, \dots w_p)$ solve for u_{jk}

2) Find the ground state of $H_{\vec{u}}$ ("fermionic GS")

$E = E(w_1, \dots, w_m)$ $|\tilde{\Psi}_{\vec{u}}\rangle \in \tilde{\mathcal{M}}$

Theorem (Lieb).

$E(W_1, \dots, W_m)$ achieves its minimum when $W_p = 1$ for all p ⟹ the ground state of the spin Hamiltonian is vortex-free.

3-5

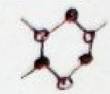
3) Project $|\tilde{\Psi}_{\vec{u}}\rangle$ onto the physical subspace, i.e. symmetrize over $\mathbb{Z}_2$-gauge transformation D_i

$$\underbrace{|\Psi_{\vec{w}}\rangle = \prod_i \left(\frac{1+D_i}{2}\right) |\Psi_{\vec{u}}\rangle}$$

Solving the quadratic Hamiltonian for the vortex-free field $\vec{u}$,

$$\underbrace{H = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k},$$

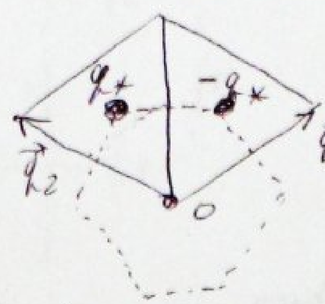
We just diagonalize iA. Two sites per unit cell $\Rightarrow$ $i\tilde{A}(q) = \begin{pmatrix} 0 & f(q) \\ f(q)^* & 0 \end{pmatrix}$



we find a one-particle spectrum for fermionic excitations

If $J_x = J_y = J_z$, this is the same as the spectrum of graphene

Momentum space



$\varepsilon(q) \sim \sqrt{\delta q_x^2 + \delta q_y^2}$

$J_x \simeq J_y \simeq J_z$ – two nodes (Dirac points) in the spectrum.

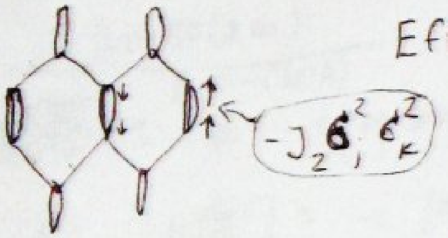
$J_z > J_x + J_y$ – the nodes merge and disappear $\Rightarrow$ the spectrum is gapped.

Vortices are always gapped.

Gapped phases: relation to the toric code

Suppose $J_x, J_y \ll J_z$

Perturbation theory



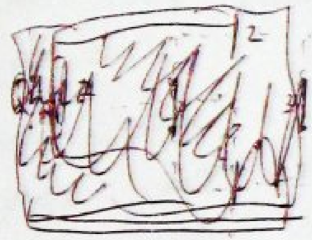
Effective spins: $\Uparrow = \uparrow\uparrow$

$\Downarrow = \downarrow\downarrow$

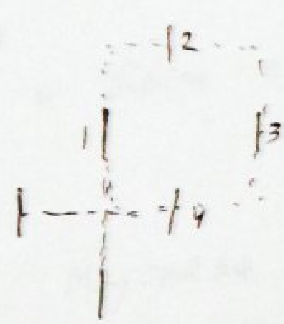
Second order: $\delta E \sim -N \dfrac{J_x^2 + J_y^2}{4 J_z}$

Fourth order:

$$H_{eff}^{(4)} = \text{const} - \frac{J_x^2 J_y^2}{16 J_z^3} \sum_p Q_p$$

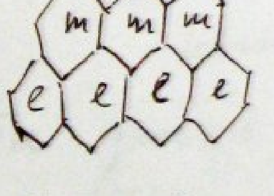


$$Q_p = \sigma_1^y \sigma_2^z \sigma_3^y \sigma_4^z$$



Permuting $\sigma_j^x, \sigma_j^y, \sigma^z$ in a certain way, we get the toric code Hamiltonian

Weak breaking of the translational symmetry



$$E_m = E_e \approx \frac{J_x^2 J_y^2}{8 J_z^3}$$

$$E_\varepsilon \approx 2 J_z \gg E_m + E_e$$

Free fermions will decay into $m \times e$ in the presence of a bath.

Quadratic fermionic Hamiltonians

$$\boxed{H(A) = \frac{i}{4} \sum_{j,k} A_{jk}\, c_j c_k}$$

real skew-symm matrix

$\underline{A \in so(2m)}$

[redacted] $A \to -iH(A)$ is a representation of $so(2m)$ in the Fock space ($\dim \mathcal{F} = 2^m$)

$[-iH(A), -iH(B)] = -i\, H([A,B])$, This is the mathematical reason why quadratic Hamiltonians are simple.

Canonical form of a skew-symm real matrix

$$A = Q \begin{pmatrix} 0 & \varepsilon_1 & & & \\ -\varepsilon & & & & \\ & & \ddots & & \\ & & & 0 & \varepsilon_m \\ & & & -\varepsilon_m & 0 \end{pmatrix} Q^T \qquad Q \in O(2m)$$

How does this translate to the second quantization language?

Definition

Fermionic mode is a linear comb. of c_j with complex coefficient

Majorana mode is a linear comb. with real coefficients

$F(x) = \sum_j x_j c_j$

If we introduce Majorana modes

$(b_1', b_1'', \ldots b_m', b_m'') = (c_1, \ldots c_{2m})\, Q$,

then
$$\boxed{H = \frac{i}{2} \sum_{k=1}^{m} \varepsilon_m\, b_k' b_k'' = \sum_{k=1}^{m} \varepsilon_k \left(\hat{a}_k^+ \hat{a}_k - \frac{1}{2}\right)}$$

$\leftarrow$ canonical form of quadratic Hamiltonian

where $a_k = \frac{1}{2}(b_k' + i b_k'')$

The ground state is defined by the condition $a_k |\Psi_0\rangle = 0$

In fact, $|\Psi_0\rangle$ is determined by a so-called <u>structural matrix</u>

$$B = -i\,\mathrm{sgn}(iA) = Q \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix} Q^T$$

B is real skew-symmetric

$B^2 = -I$

For instance,

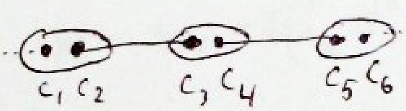
$$\langle \Psi_0 | c_j c_k | \Psi_0 \rangle = \delta_{jk} + i B_{jk}$$

In many cases, the spectrum is not important. It is often enough to know that it is gapped. The structural matrix is most important from the mathematical perspective.

Theorem: If Spec(A) is gapped and A only involves matrix elements between neighbors, then B is quasidiagonal:

$$|B_{jk}| < C \exp\left(-\frac{|j-k|}{3}\right)$$

<u>Unpaired Majorana modes</u>



$$H = \frac{i}{2} \sum_j c_{2j} c_{2j+1} = \frac{1}{2} \sum_j (a_j + a_j^\dagger)(a_{j+1} - a_{j+1}^\dagger)$$

$a_1, a_1^\dagger$ (spinless fermionic mode)

There is gap in the bulk.

Dangling bonds at the ends of the chain $\Rightarrow$ $\Rightarrow$ zero modes.

This is a topological phenomenon.

We can define B_{jk} in the bulk but cannot extend in to a neighborhood of the boundary while preserving the condition $B_{jk}^2 = -I$

<u>Topological obstruction</u>

(I call it "cutting obstruction": B is well-defined on an inf. chain but cannot be defined if we cut the chain and require that there are no cross-terms

Theory of quasidiagonal matrices: noncommutative geometry for pedestrians

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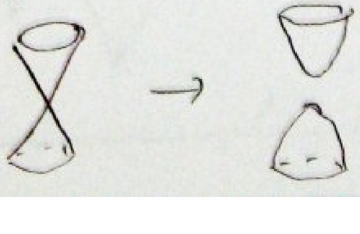
<u>Now, let us return to the honeycomb lattice model and apply a magnetic field.</u>

$$H = H_0 - \sum_j \left(h_x \sigma_j^x + h_y \sigma_j^y + h_z \sigma_j^z\right) \qquad (J_x = J_y = J_z = J)$$

Third order of perturbation theory:

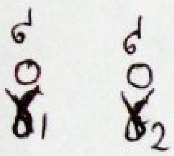
$$i\tilde{A}(q) = \begin{pmatrix} \Delta(q) & f(q) \\ f(q)^* & -\Delta(q) \end{pmatrix} \qquad \Delta \sim \frac{h_x h_y h_z}{J^2}$$

<u>Dirac points</u>  massive Dirac fermions

The two-dimensional structural matrix B (or the spectral projector $P = \frac{1}{2}(1 - iB)$) is characterized by a Chern number

$$\underbrace{\nu = \operatorname{sgn} \Delta = \pm 1}$$

Theorem If ν is odd, then vortices carry unpaired Majorana modes



$a = \frac{1}{2}(\gamma_1 + i\gamma_2)$

Two states : $a^+ a |\Psi_0\rangle = 0$
$a^+ a |\Psi_\varepsilon\rangle = 1$

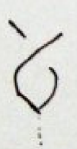
vortex fermion

$$\boxed{\begin{aligned} \sigma \times \sigma &= 1 + \varepsilon \\ \sigma \times \varepsilon &= \sigma \\ \varepsilon \times \varepsilon &= 1 \end{aligned}}$$

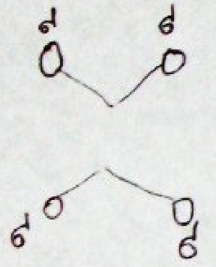
fusion rules for non-Abelian anyons (for any odd ν)

Braiding rules depend on ν mod 16

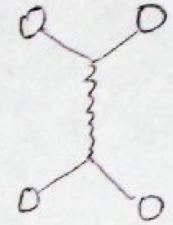
(Of course, to have $|\nu| > 1$, one needs a more general model).

For $\nu = 1$  $= e^{-i\frac{\pi}{8}}$  ,  $= e^{i\frac{3\pi}{8}}$ 

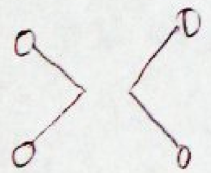
Anyonic qubit (four vortices)



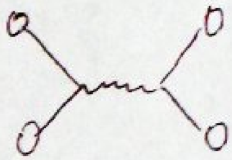
$|0\rangle$



$|1\rangle$



$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$



$$= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

associativity relations