PCE STAMP: LECTURE 4

MORE ABSTRACT MODELS of DECOHERENCE:

QUANTUM WALKS

&

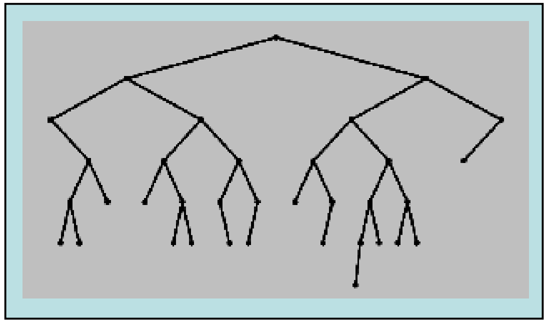
TOPOLOGICAL FIELD THEORIES

4.1: QUANTUM WALKS & QUANTUM INFORMATION

We now want to look in a more general way at the whole issue of large scale quantum effects (interference, entanglement, coherence, etc.) and the influence of quantum fluctuations and decoherence on these.

The first way we shall do this is by looking at 'quantum walks', and the influence of decoherence on these. Quantum walks refer to the dynamics of a particle on some arbitrary mathematical graph. Their importance is twofold. First, they can be mapped to a very large class of quantum information processing systems. Second, they can be used to generate new kinds of quantum information processing algorithm. The whole field of quantum walks is rather new, and there is still elementary basic work to be done. One of the most interesting things is the investigation of decoherence on quantum walks. Here I briefly describe some recent results in this field, and their implications for large-scale quantum phenomena

Remarks on NETWORKS- the QUANTUM WALK



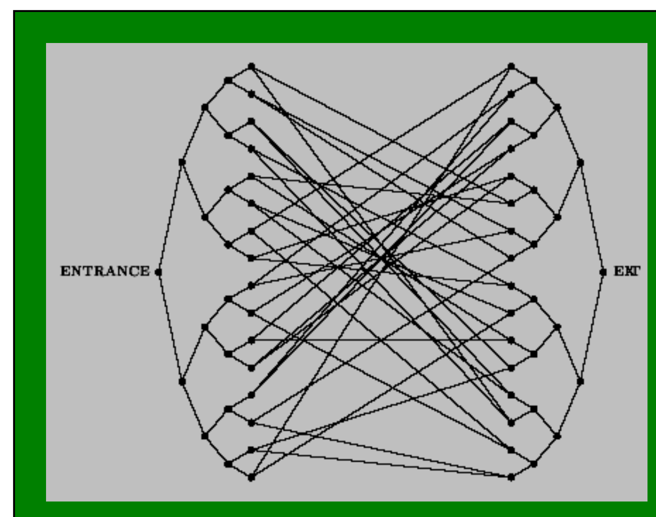
Computer scientists have been interested in RANDOM WALKS on various mathematical GRAPHS, for many years. These allow a general analysis of decision trees, search algorithms, and indeed general computer programmes (a Turing machine can be viewed as a walk). One of the most important applications of this has been to error correction- which is central to modern software.

Starting with papers by Aharonov et al (1994), & Farhi & Gutmann (1998), the same kind of analysis has been applied to QUANTUM COMPUTATION. It is easy to show that many quantum computations can be modeled as QUANTUM WALKs on some graph. The problem then becomes one of QUANTUM DIFFUSION on this graph, and one easily finds either power-law or exponential speed-up, depending on the graph. Great hopes have been pinned on this new development- it allows very general analyses, and offers hope of new kinds of algorithm, and new kinds of quantum error correction- and new ‘circuit designs’.

Thus we are interested in simple walks described by Hamiltonians like

$$\hat{H} = -\sum_{\langle ij \rangle} \Delta_{jk} \left(\hat{c}_i^\dagger \hat{c}_j + \hat{c}_i \hat{c}_j^\dagger \right) + \sum_{k=0}^{2^N} V_k \hat{c}_k^\dagger \hat{c}_k$$

which can be mapped to a variety of gate Hamiltonians, spin Hamiltonians, and interacting qubit networks. Most of all we want to understand how decoherence affects the quantum walk dynamics; ie., we couple oscillator and spin baths to the walker.



A VARIETY of MAPPINGS

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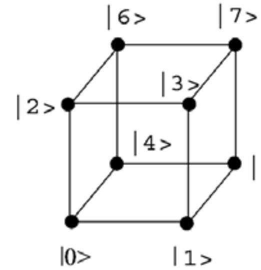
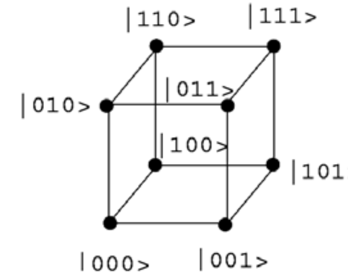
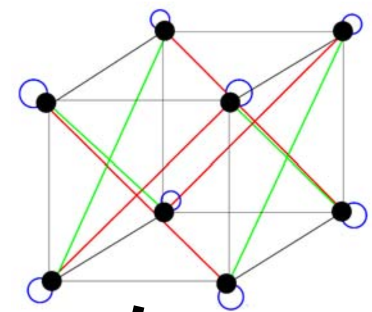
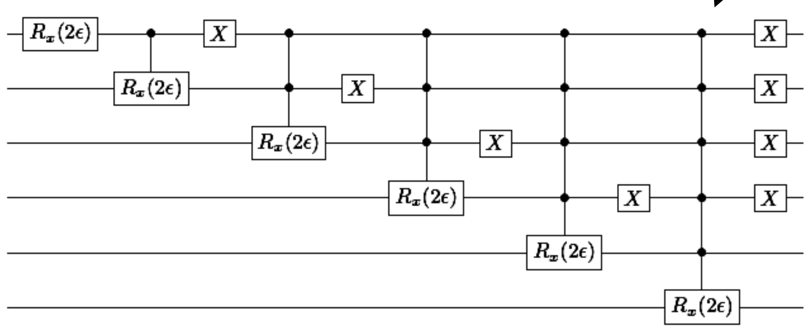
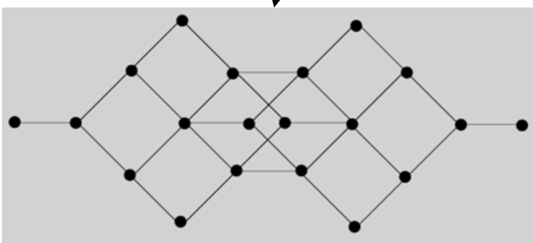
One can make a lot of useful mappings between qubit Hamiltonians, Hamiltonians for spin chains and other spin networks, quantum gate systems, and quantum walk Hamiltonians. This is very useful in the exploration of different quantum algorithms and quantum information processing hierarchies.



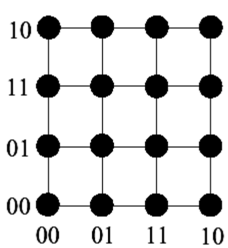








One of the most important goals in this field is to try and produce new kinds of quantum algorithm. So far the 2 most important ones are the Shor and Grover algorithms. The hope is that new representations, like the quantum walk, will allow us to do this.



Another important use of quantum walks is the possibility of more easily studying decoherence in different quantum information processing systems. The mappings above allow us to easily move between different representations of this, and to easily study the dynamics of quantum information processing.

Initial results by various groups have been very interesting – they have helped clarify the conditions under which one gets accelerated dynamics (notably on hyperlattices like the one at left).

DECOHERENCE & QUANTUM WALKS – a MODEL EXAMPLE

NV Prokof'ev, PCE Stamp
condmat/0605097

To see the usefulness of the analysis of decoherence, let's look at a simple but interesting example. We look at a d-dimensional hyperlattice, and add a transverse coupling to a spin bath:

$$\mathcal{H} = \Delta_o \sum_{<ij>} \left\{ c_i^\dagger c_j \cos\left(\sum_k \alpha_k \sigma_k^x \right) + H.c. \right\}$$

FREE QUANTUM BEHAVIOUR

Suppose initial state is at origin: $\psi\mathbf{n}(t = 0) = \delta_{\mathbf{n}0}$ So that: $\psi_{\mathbf{n}}(t) = L^{-d} \sum_{\mathbf{k}} e^{i\mathbf{kn} - i\epsilon(\mathbf{k})t}$

Then, since $P^0_{\mathbf{n}}(t) = |\psi_{\mathbf{n}}(t)|^2$ One gets $P^0_{\mathbf{n}}(t) = \prod_{\mu=1}^{d} J^2_{n_\mu}(z)\ ; \quad z = 2\Delta_o t$

More generally, we can start with a wave-packet: $\psi_{\mathbf{n}}(t=0) \approx \left(\frac{1}{\sqrt{\pi} R} \right)^{d/2} e^{-n^2/2R^2}$

which gives $P^0_{\mathbf{n}}(t) \approx \left(\frac{R^2}{\pi(R^4 + z^2)} \right)^{d/2} e^{-n^2 R^2/(R^4+z^2)}$

Thus, quite generally one has $P^0_0(t) \propto 1/t^d$ and that $\langle R^2 \rangle \propto t^2$

Now this is to contrasted with diffusive behaviour:

$P^{(cl)}_0(t) \propto 1/t^{d/2}$ and $\langle R^2 \rangle \propto t$

DECOHERENCE DYNAMICS

For the decoherent Quantum Walk Hamiltonian

$$P_{\mathbf{n}}(t) = \int_0^{2\pi} \frac{d\varphi}{2\pi} \prod_{\mu=1}^{d} J_{n_\mu}^2(z\cos\varphi)$$

Or, for an initial wave-packet

$$P_{\mathbf{n}}(t) \approx \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{R^d\, e^{-n^2R^2/(R^4+z^2\cos^2\varphi)}}{\left[\pi(R^4+z^2\cos^2\varphi)\right]^{d/2}}$$

Now this produces a very surprising result:

$$P_0(z\to\infty) \approx \frac{A_d R^{2-d}}{\Delta_o t}$$

BUT....

$$< ((\mathbf{n}(t)-\mathbf{n}(0))^2 >= 12\sum_n \mathbf{n}^2 P_{\mathbf{n}}(z) = \frac{d}{2}(\Delta_o t)^2$$

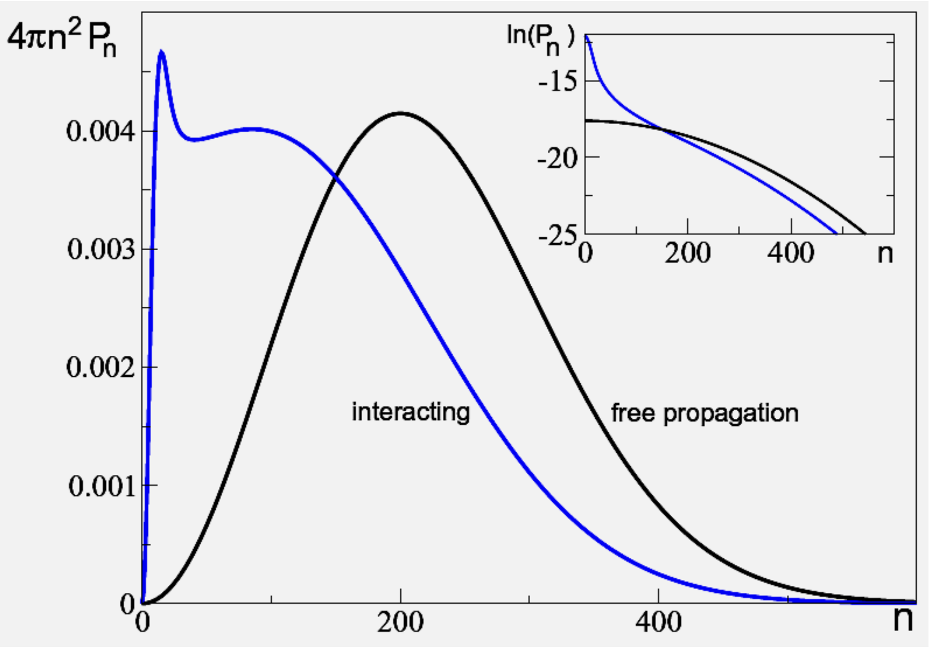
In other words, the particle spends more time near the origin than classical diffusion would predict, BUT it also has a BALLISTIC part (in the long-time limit)!!

More detailed evaluation of the integrals fills this picture out:

$$P_{\mathbf{n}}(t) \approx \frac{2}{\pi^{d/2+1}R^d}\int_0^\infty d\varphi\, \frac{e^{-r^2/u^2\varphi^2}}{u^d\varphi^d}$$

$$= \frac{\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{d/2+1}}\,\frac{R}{\Delta_o t\, n^{d-1}}\,, \qquad (R \ll n \ll \Delta_o t/R)$$

$$P_{\mathbf{n}\to\infty}(t) \approx \frac{1}{\pi^{(d+1)/2}n}\left(\frac{R}{2\Delta_o t}\right)^{d-1} e^{-n^2(R/2\Delta_o t)^2}$$



Density matrix after time t such that $z=2\Delta t >> R^2$, with z = 2000 and R=10. Long-range part is ballistic, short-range part is sub-diffusive.

2.2: Analysis of a Topological Field Theory

The
DISSIPATIVE W.A.H. MODEL

We are interested in topological field theories because they possess ‘hidden’ topological quantum numbers which are conserved even when the system is subject to quite severe perturbations. A model of central interest is the ‘dissipative W.A.H. model’ (named after Wannier, Az’bel, & Hofstadter’). This is produced by synthesizing elements from 2 simpler models which are very interesting on their own: the W.A.H. model (non-interacting charged particles moving on a 2-d lattice in a uniform magnetic field), and the ‘Schmid model’ (a particle moving in a periodic potential, coupled to an oscillator bath). Combining these gives a model in which W.A.H. particles couple dissipatively to an oscillator bath. This model is believed to have an SL(2,Z) symmetry, in common with some other field theories which attempt to describe the Fractional Quantum Hall liquid, certain systems of interacting quantum wires, and possibly Josephson junction arrays. However the model was actually originally introduced by string theorists (Callan et al.) to deal with a class of open string theories, and it is still of central interest in string theory. It is of potential interest for topological quantum computation.

In what follows I first describe key results for the W.A.H. and Schmid models on their own, and then go on to discuss results for the dynamics of the dissipative WAH model. It is found that there are some important outstanding problems here – in particular, the older results of the string theorists seem to conflict with more recent results.

The W.A.H. MODEL

M. Azbel, Zh. Eksp. Teor. Fiz. 46, 929 (1964) [Sov. Phys. JETP 19, 634 (1964)]; D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976); G. H. Wannier, G. M. Obermair, and R. Ray, Phys. Status Solidi (b) 93, 337 (1979).

$$H=\frac{1}{2m}\left[-i\hbar\nabla-\frac{e}{c}\mathbf{A}\right]^2+V(\mathbf{x})$$

$$V(\mathbf{x}+\mathbf{l})=V(\mathbf{x}) \qquad \mathbf{l}=n\,\mathbf{e}_1+m\,\mathbf{e}_2$$

The Hamiltonian involves a set of charged fermions moving on a periodic lattice- interactions between the fermions are ignored. The charges couple to a uniform flux through the lattice plaquettes.

Often one looks at a square lattice, although it turns out much depends on the lattice symmetry.

$$V=-V_0[\cos(2\pi x/a)+\cos(2\pi y/a)-2]$$
$$=V_0[\cos(k_0\hat{x})+\cos(k_0\hat{y})]$$

One key dimensionless parameter in the problem is the FLUX per plaquette, in units of the flux quantum

$$\alpha=a^2Be/hc$$

The crucial effect of the applied field is in the extra phase accumulated around each lattice plaquette- these phases of course interfere with each other. To see this we choose the Landau gauge:

$$\vec{A} = H(0, x, 0)$$

Then , wrting the Schrodinger eqtn as a difference eqtn. around a plaquette, we have:

$$E_0[\psi(x+a, y) + \psi(x-a, y) + e^{-ieHax/\hbar c}\psi(x, y+a) + e^{+ieHax/\hbar c}\psi(x, y-a)] = E\psi(x, y)$$

In terms of reduced variables (where E_0 is just the bandwidth) we can then write the solution in the form:

$$x = ma, \quad y = na, \quad E/E_0 = \epsilon$$

$$\psi(ma, na) = e^{i\nu n} g(m)$$

The Schrodinger eqtn. takes the iterative form:

$$\begin{pmatrix} g(m+1) \\ g(m) \end{pmatrix} = \begin{pmatrix} \epsilon - 2\cos(2\pi m\alpha - \nu) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g(m) \\ g(m-1) \end{pmatrix}$$

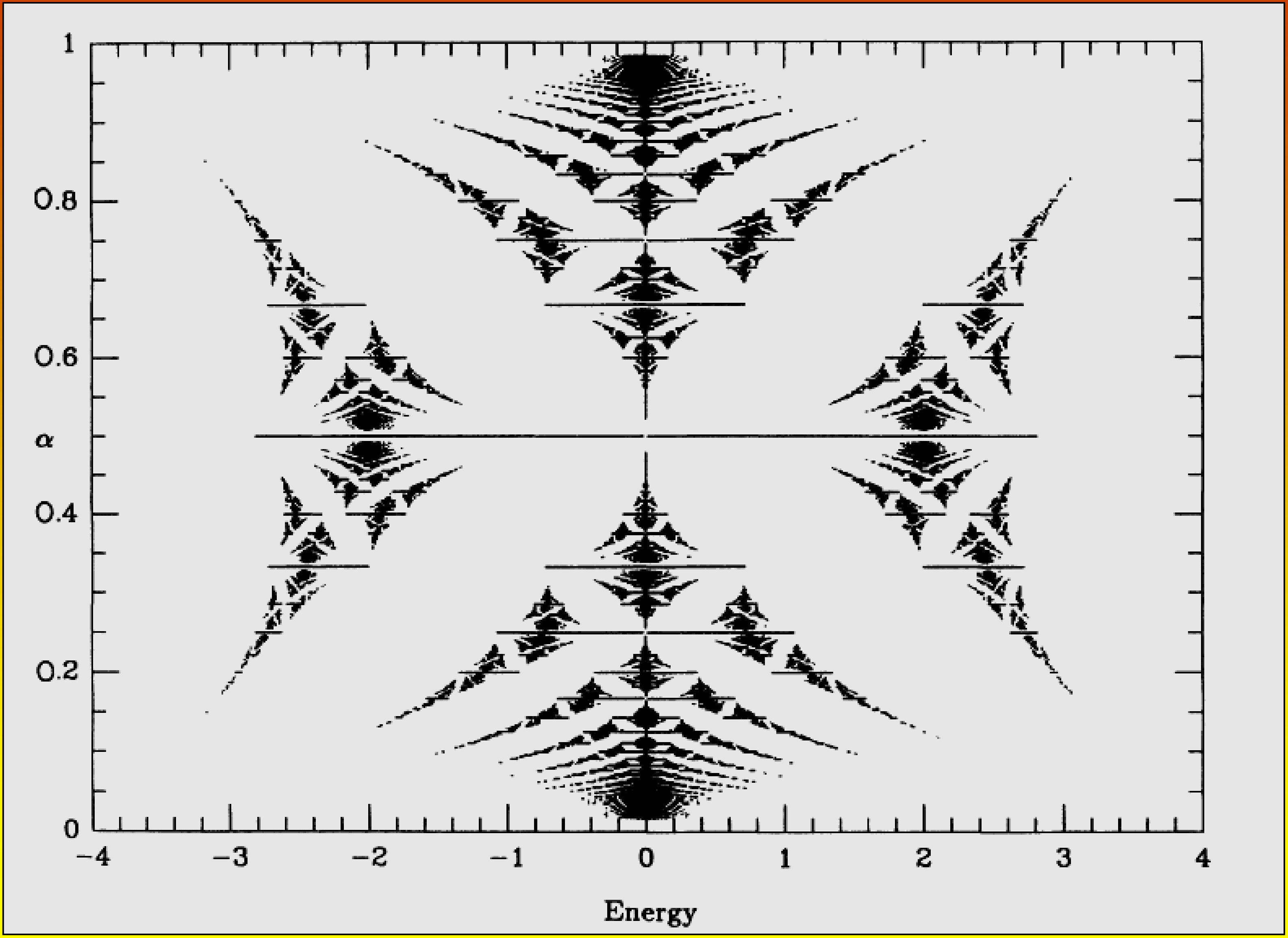
Solutions exist provided:

$$2\pi\alpha(m+q) - \nu = 2\pi\alpha m - \nu + 2\pi p$$

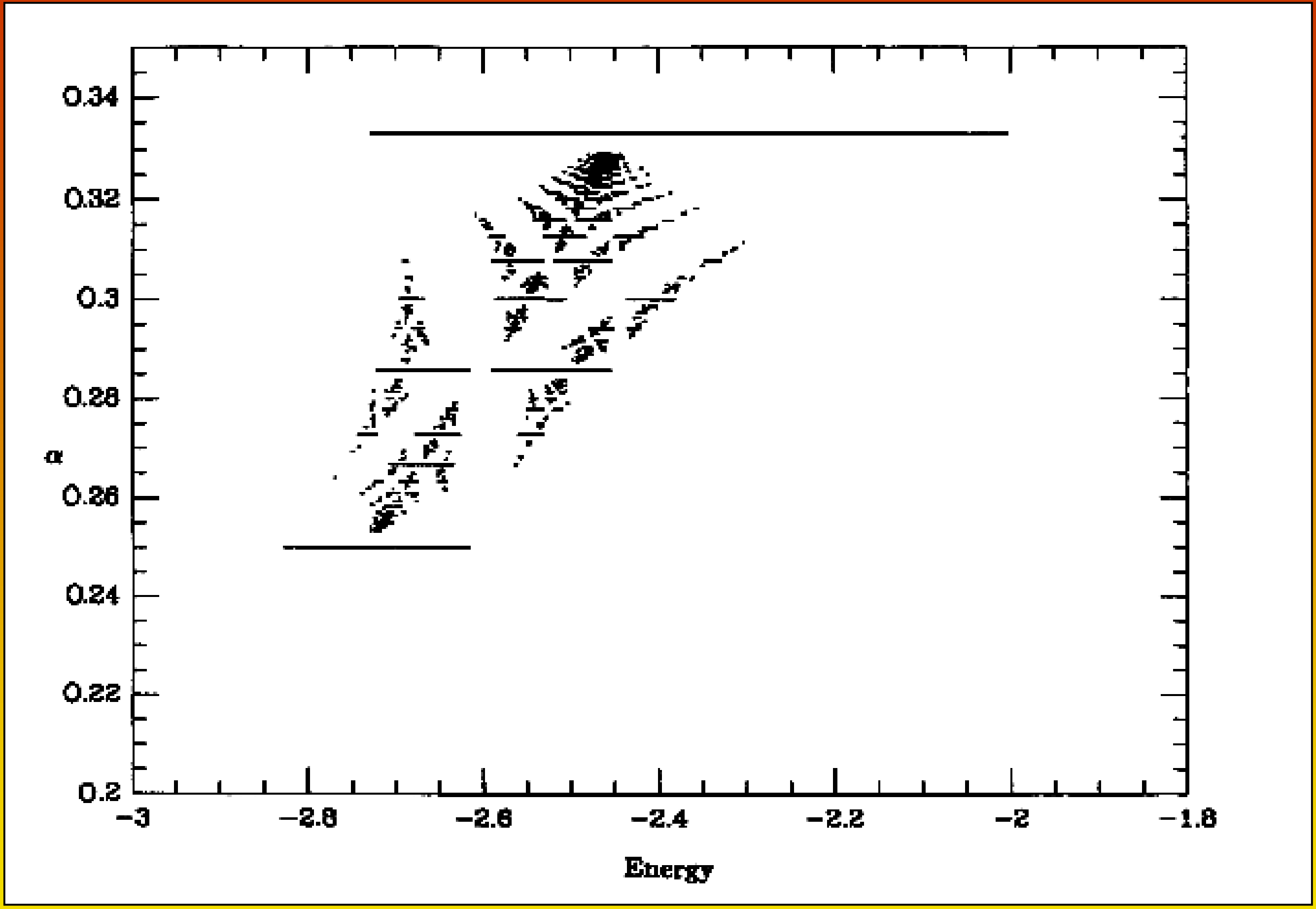
This is just a condition on the flux- it must be rational:

$$\alpha = p/q$$

The HOFSTADTER BUTTERFLY

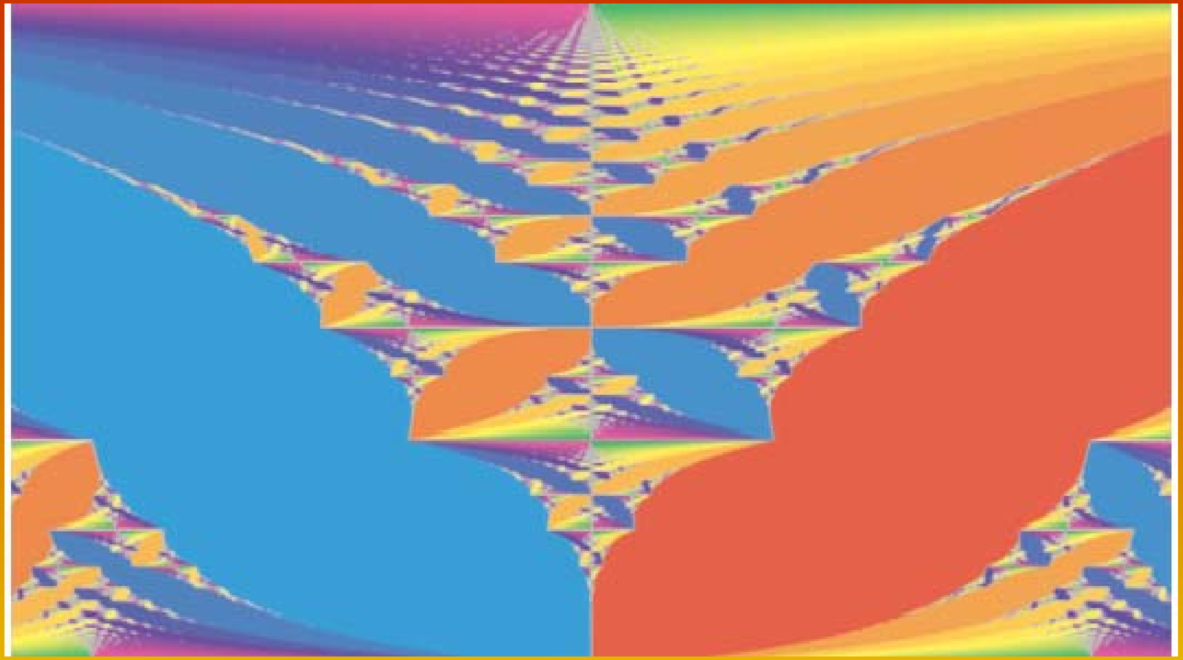


The graph shows the 'support' of the density of states- provided α is rational



The recursive nature of the Schrodinger eqtn is then directly responsible for the recursive form of the density of states. One has a nested pattern in which the entire form is repeated in any subset (reflecting of course the structure of the rational numbers). The 'shape' of the nesting pattern depends on the lattice structure. For a finite lattice the adjustment of levels between very close values of flux is effected by level crossing between band edges. For infinite lattices this happens infinitely fast. In finite lattices, EDGE states are crucial.

Another way of looking at W.A.H. (Chern #)



D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982)

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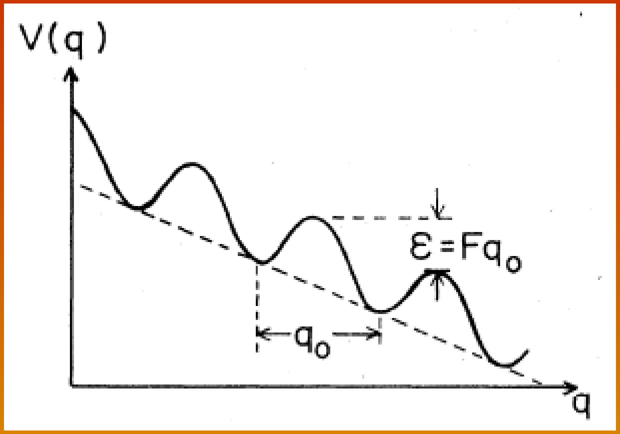
II: Schmid model- particle coupled to oscillator bath



In the Schmid model a particle moves on a 1-d periodic lattice, but is now coupled to an oscillator bath. It is then interesting to apply a weak field.

$$\hat{\mathcal{H}} = \frac{P^2}{2M} + V(q) + \sum_\alpha P_\alpha^2/2m_\alpha + \frac{1}{2}\sum_\alpha m_\alpha \omega_\alpha^2 (x_\alpha + q\lambda_\alpha/m_\alpha\omega_\alpha^2)^2$$

The quantity of crucial interest is then the particle mobility (hopefully well-defined!).

$$V(q) = -V\cos(2\pi q/q_0) - Fq$$

The particle-bath interaction is bilinear in the coordiantes of the two. The individual couplings are weak (delocalised modes), but their cumulative effect on the particle depends on the form of the Feynman-Vernon/Caldeira-Leggett 'spectral density', defined as follows:

$$J(\omega) = \frac{\pi}{2}\sum_\alpha \frac{\lambda_\alpha^2}{m_\alpha \omega_\alpha}\delta(\omega - \omega_\alpha);$$

In this study we choose an 'Ohmic' spectral form:

$$J(\omega) = \eta\omega$$

The Schmid model is a very rich field theory. We first separate the action into 2 terms:

$$S_{eff} = S_0 + S_{int}$$

The 'bare' action contains the interactions with the bath- this is the Caldeira-Leggett action:

$$S_0 = \int d\tau \frac{1}{2} m \dot{q}_\tau^{\,2} + \frac{\eta}{4\pi} \int d\tau\, d\tau' \left(\frac{q_\tau - q_{\tau'}}{\tau - \tau'} \right)^2$$

The 'interaction' term is the periodic potential:

$$S_{int} = \int d\tau\, V(q_\tau)$$

The reasons for making this choice will become clear.

The bare action is a simple quadratic form:

$$S_0 = \tfrac{1}{2} \int (d\omega/2\pi) D_\omega^{-1} |q_\omega|^2$$

The propagator describes quantum Brownian motion:

$$D_\omega^{-1} = m\omega^2 + \eta|\omega|$$

$$D_{\tau\tau'} = \int (d\omega/2\pi) D_\omega \exp - i\omega(\tau - \tau') = \begin{cases} -(1/2\eta)r, & r \ll 1, \\ -(1/\pi\eta)\ln r, & r \gg 1 \end{cases}$$

$$r = (\eta/m)|\tau - \tau'|$$

A crucial feature of the Ohmic form is that we get a logarithmic interaction generated between states of the particle separated by long time intervals- leading to IR divergence at low energy in the particle dynamics.

To understand this model we start with the partition function, written as a path integral over trajectories

$$Z[F_\tau] = \int \mathfrak{D}q_\tau \exp(-S_{eff})$$

(1) EXPANSION in POTENTIAL

Let us assume that we can expand in g:

$$V(q_\tau) = -g\cos q_\tau + F_\tau q_\tau$$

In the action we easily get:

$$\exp(g\int d\tau \cos q_\tau) = \sum_{n=0}^{\infty} \int d\tau_1 \ldots d\tau_n \frac{(g/2)^n}{n!} \sum_{\{e_j\}} \exp[i\int d\tau \sum_{j=1}^{n} e_j \delta(\tau-\tau_j) q_\tau]$$

$$e_j = \pm 1$$

We can consider the $\{e_j\}$ to be classical charges located at $\{\tau_j\}$. We deal with the standard 'Coulomb gas'; the partition function is only well-defined if the system is globally neutral. We give it a charge density

$$\rho_\tau^N = \delta(\tau-\tau_1) + \ldots + \delta(\tau-\tau_N) - \delta(\tau-\tau_{N+1}) - \ldots - \delta(\tau-\tau_{2N})$$

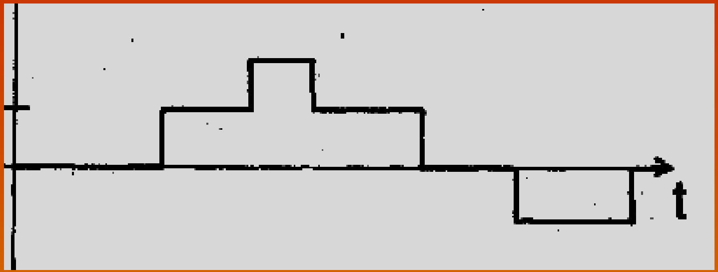
We then have:

$$Z = C\sum_N \int d\tau_1 \ldots d\tau_{2N} [(g/2)^N/N!]^2 \exp\{-\tfrac{1}{2}\int d\tau\, d\tau' [iF_\tau + \rho_\tau^N] D_{\tau\tau'} [iF_{\tau'} + \rho_{\tau'}^N]\}$$

with the usual normalisation $C = (\det D)^{-1/2}$

(2) DUAL INSTANTON EXPANSION

The WKB/instanton expansion is valid in the regime where $s = 8(mg)^{1/2} \gg 1; \quad s \gg \eta.$



The particle then tunnels between wells through large barriers- this is the large g limit. We can write the trajectory in the form at right:

$$\hat{q}_\tau = \sum_{j=1}^{n} e_j f(\tau - \tau_j)$$

$$f(\tau) = 4 \arctan \exp(\omega_0 \tau)$$

We then have an action with interactions between local 'instanton charges':

$$S_{\text{eff}}[\hat{q}] = ns + \tfrac{1}{2}\sum_{jk} e_j e_k \Delta_{\tau_j \tau_k} + i\int (d\omega/2\pi)(F_{-\omega} h_\omega/\omega) \sum_j e_j \exp(i\omega\tau_j)$$

Again, we require global charge neutrality: $\sum e_j = 0$

Again, we have long-range log interactions:

$$\Delta_{\tau\tau'} = \int (d\omega/2\pi) h^2(\omega)(\eta/|\omega|) \exp - i\omega(\tau - \tau') = \begin{cases} -\alpha\eta\rho^2, & \rho \ll 1, \\ -4\pi\eta\ln\rho, & \rho \gg 1 \end{cases}$$

(here, ω_o is the 'bounce frequency', and $\rho = \omega_0 |\tau - \tau'|$

DUALITY

We now see that the duality can be written: $D_{\tau\tau'} \to \Delta_{\tau\tau'}$

provided we make the following change: $g \to \gamma$ $\quad \gamma/2 = \omega_0 (2s/\pi)^{1/2} e^{-s}$

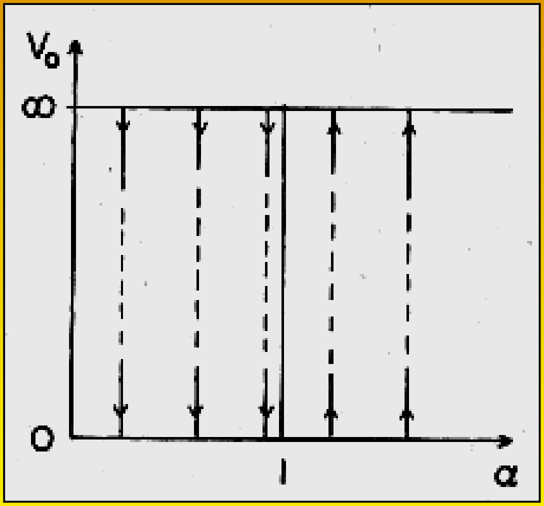
PHASE DIAGRAM

This system is governed by the extremely well-known 'Kosterlitz-Thouless' scaling. The 2 phases differ in the 'mobility' of the particle, defined in terms of the partition function by $\mu(\omega) = \eta |\omega| \langle qq \rangle_{\omega}$ $\quad \langle q_\tau q_{\tau'} \rangle = Z^{-1} \delta^2 Z / \delta F_\tau \delta F_{\tau'} |_{F=0}$

The KT scaling theory then shows that one has a localised phase at T = 0 when $\alpha > 1$, and a delocalised phase when $\alpha < 1$. This general conclusion can also be arrived at by direct calculation.

The duality appears in the mobility in the following form: $\mu \to 1 - \mu$ $\quad 2\pi\eta \to 1/2\pi\eta$

$mg/\eta \to \gamma/\omega_0$



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This model has recently been re-evaluated, with some interesting exact solutions:
M Hasselfield, T Lee, GW Semenoff, PCE Stamp: hep-th/0512219 (Ann Phys, in press)

III: W.A.H. + Boson Field/ Oscillator bath/gauge fluct^{ns}

So now we arrive at the model we really want to study. This problem is produced by combining the 2 previous problems- we have a 2-d WAH lattice with particles coupled to an oscillator bath:

$$H = \frac{1}{2m}\left[-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right]^2 + V(\mathbf{x}) + \sum_\alpha P_\alpha^2/2m_\alpha + \frac{1}{2}\sum_\alpha m_\alpha\omega_\alpha^2(x_\alpha + q\lambda_\alpha/m_\alpha\omega_\alpha^2)^2$$

$$V = -V_0[\cos(2\pi x/a) + \cos(2\pi y/a) - 2]$$
$$= V_0[\cos(k_0\hat{x}) + \cos(k_0\hat{y})]$$

There are now TWO dimensionless couplings in the problem- to the external field, and to the bath:

$$2\pi\alpha = a^2\eta/\hbar$$

$$2\pi\beta = \frac{eB}{\hbar c}a^2$$

The effective Hamiltonian is also written as:

$$H = -t \sum_{ij} [c_i c_j \exp\{iA_{ij}\} + H.c.] \quad \text{....... “WAH” lattice}$$

$$+ \sum_n \sum_q \lambda_q R_n . x_q \quad + \quad H_{osc}(\{x_q\}) \quad \text{...... coupled to oscillators}$$

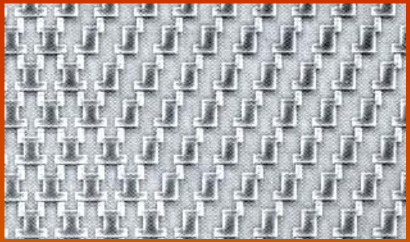
(i) the the WAH (Wannier-Azbel-Hofstadter) Hamiltonian describes the motion of spinless fermions on a 2-d square lattice, with a flux ϕ per plaquette (coming from the gauge term A_{ij}).

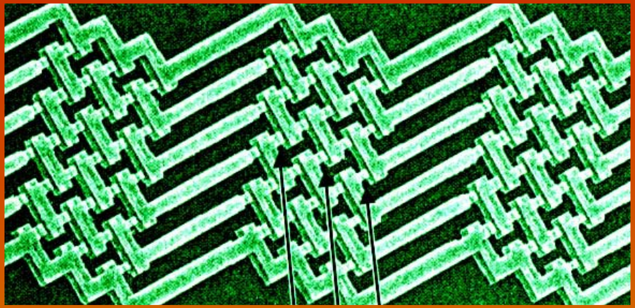
(ii) The particles at positions R_n couple to a set of oscillators.

This can be related to many systems- from 2-d J. Junction arrays in an external field to flux phases in HT_c systems, to one kind of open string theory. It is also a model for the dynamics of information propagation in a QUIP array, with simple flux carrying the info.

There are also many connections with other models of interest in mathematical physics and statistical physics.

EXAMPLE:
Superconducting arrays





The bare action is:

$$S[\phi] = \int_0^\beta d\tau \left\{ \frac{C_0}{8e^2} \sum_i (\dot\phi_i)^2 + \frac{C}{8e^2} \sum_{\langle ij \rangle} (\dot\phi_i - \dot\phi_j)^2 - E_J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j) \right\}$$

Plus coupling to Qparticles, photons, etc:

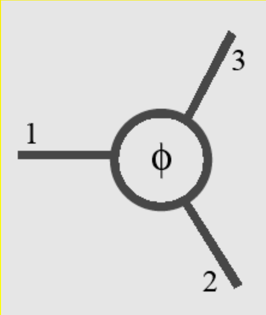
$$S_D[\phi] = \frac{1}{2} \int_0^\beta d\tau d\tau' \sum_{<ij>} \alpha(\tau - \tau') F(\phi_{ij}(\tau) - \phi_{ij}(\tau'))$$

Interaction kernel (shunt resistance is R_N):

$$\alpha(\tau) = \frac{\pi}{2e^2 R_N} \frac{1}{\beta^2} \frac{1}{\sin^2(\pi\tau/\beta)}$$

$$F_N(\phi_{ij}(\tau) - \phi_{ij}(\tau')) = \frac{1}{2} \left(\frac{\phi_{ij}(\tau) - \phi_{ij}(\tau')}{2} \right)^2$$

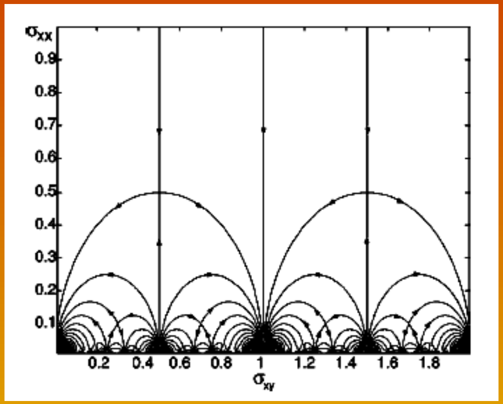
Another EXAMPLE: 3-wire junctions

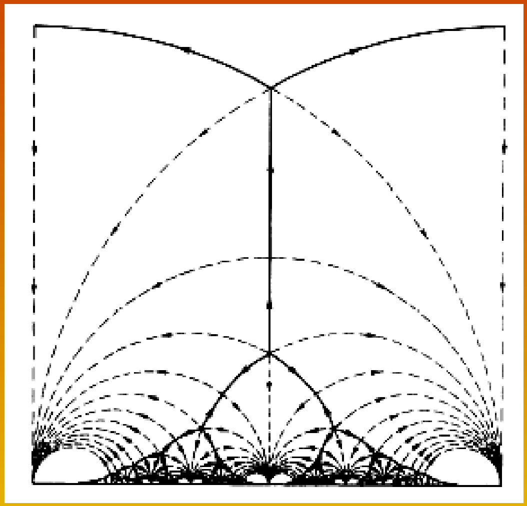


$$H_{\mathcal{B}} = -\sum_{j=1}^{3} [(\Gamma e^{i\phi/3} \psi_j^\dagger \psi_{j-1} + \text{h.c.}) + r \psi_j^\dagger \psi_j]$$

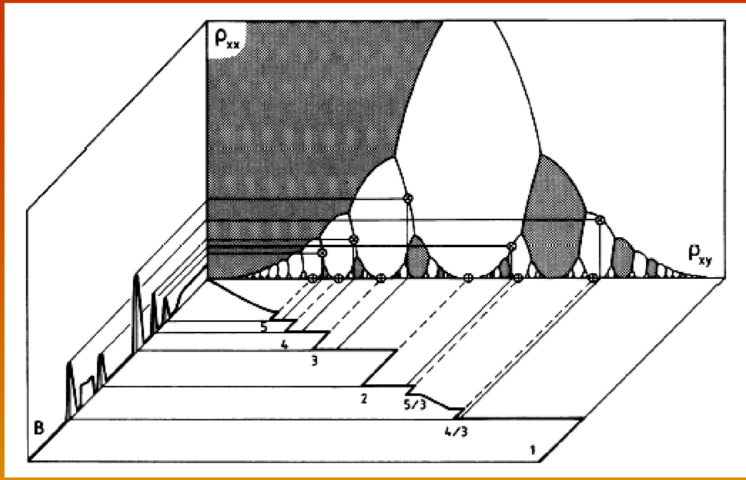
C. Chamon, M. Oshikawa, I Affleck, PRL 91, 206403 (2004)

Another EXAMPLE: FQHE

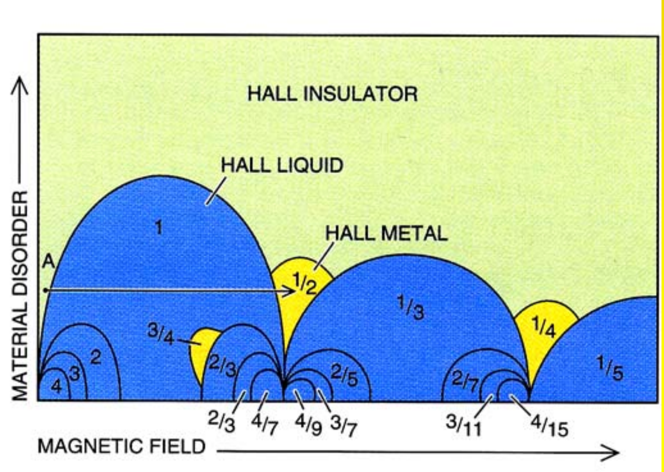




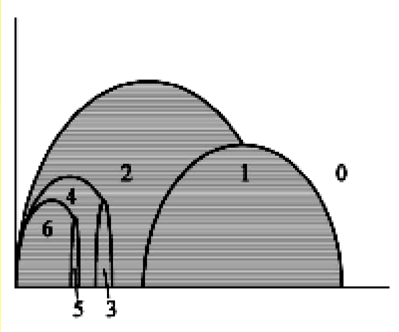
RG flow (Laughlin (1984); Lutken & Ross (1992-4))



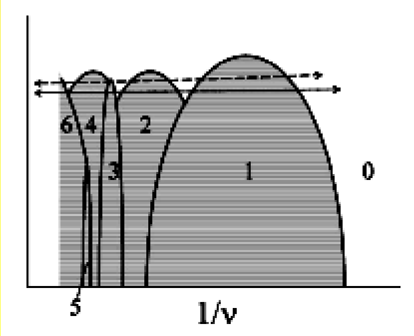
Resulting Phase diagram (Lutken & Ross (1993)



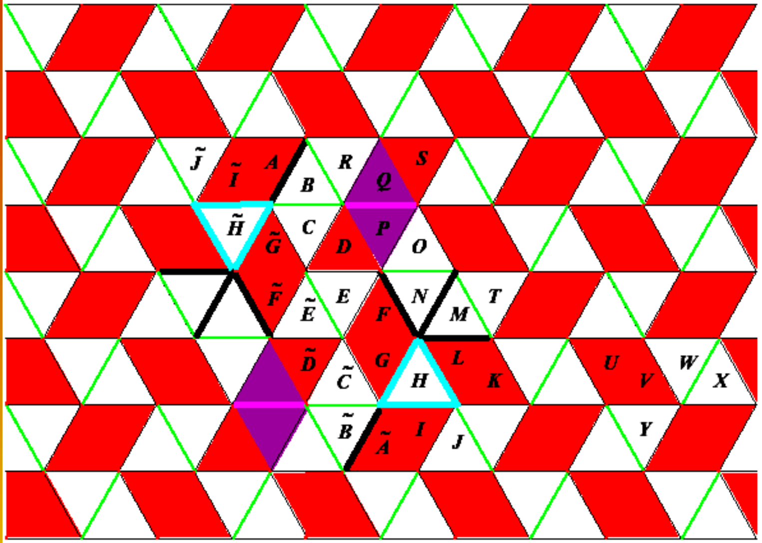
Another phase diagram (Zhang, Lee, Kivelson (1994)



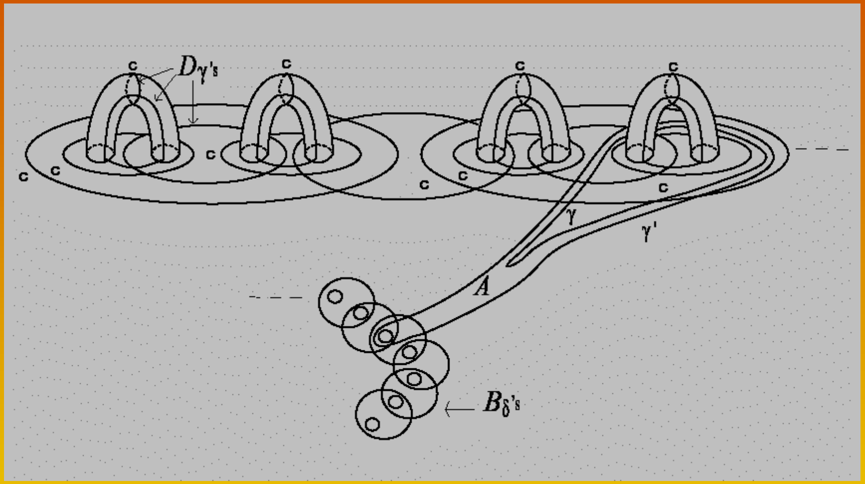
Expt (Kravchenko, Coleridge,..)



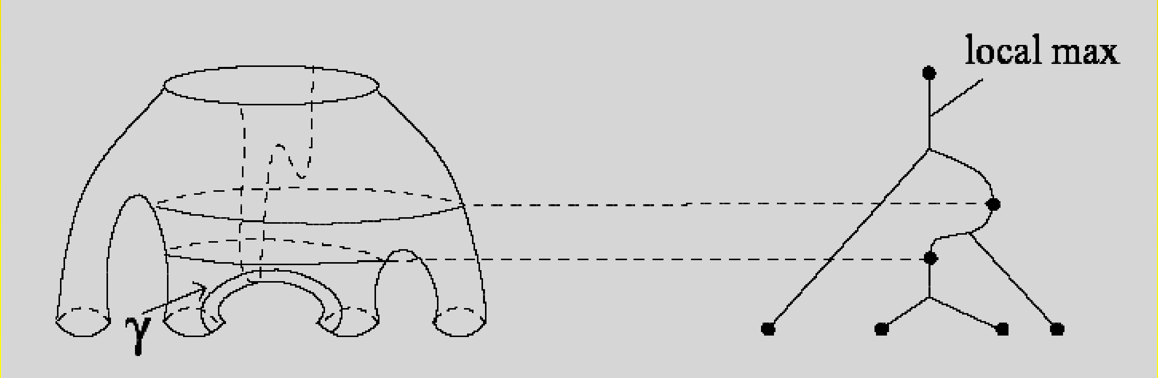
The TOPOLOGICAL QUANTUM COMPUTER



Kitaev, 1997
Freedman et al (2003, 2004)



Basic idea is to try and construct some lattice realisation of an anyon system, & use the anyons to do quantum computation. The preliminary theory indicates almost no decoherence



Problem is that so far, the only realisations of this involve very strange spin Hamiltonians- which can only be analysed using topological methods.

ACTION for the DISSIPATIVE WAH MODEL

The action is an obvious generalisation:

$$S = S_q + S_\eta + S_V$$

$$S_q = \int_{-T/2}^{T/2} dt \left[\frac{M}{2} \dot{\vec{x}}^2 + \frac{ieB}{2c} (\dot{x}y - \dot{y}x) \right]$$

$$S_\eta = \frac{\eta}{4\pi} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} dt\, dt' \left(\frac{\vec{x}(t) - \vec{x}(t')}{t - t'} \right)^2$$

$$S_V = - \int_{-T/2}^{T/2} \left[V_0 \cos\left(\frac{2\pi x(t)}{a} \right) + V_0 \cos\left(\frac{2\pi y(t)}{a} \right) \right]$$

The propagator now has a typical 'Quantum Hall' form:

$$D_{ij}(t - t'; z) = - \frac{\alpha}{\alpha^2 + \beta^2} \log(t - t')^2 \, \delta_{ij} - i \frac{\pi\beta}{\alpha^2 + \beta^2} \operatorname{sign}(t - t') \epsilon_{ij}.$$

$$\tilde{D}_{ij}(\omega; z) = \frac{1}{|\omega|} M_{ij}(\omega; 1/z),$$

$$M_{ij}(\omega, x + iy) = x\delta_{ij} + y \frac{|\omega|}{\omega} \epsilon_{ij}.$$

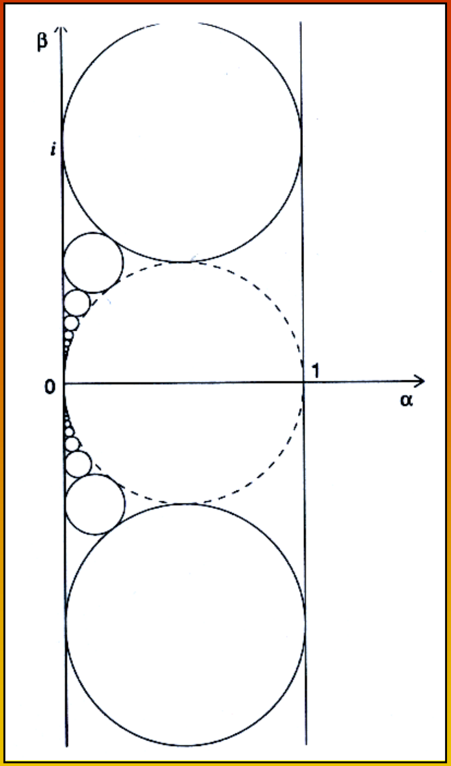
M_{ij} satisfies the following relations:

$$M_{ij}(\omega; z_1) + M_{ij}(\omega; z_2) = M_{ij}(\omega; z_1 + z_2),$$

$$M_{ij}(\omega; z_1) M_{ij}(\omega; z_2) = M_{ij}(\omega; z_1 z_2),$$

$$cM_{ij}(\omega; z_1) = M_{ij}(\omega; cz_1)$$

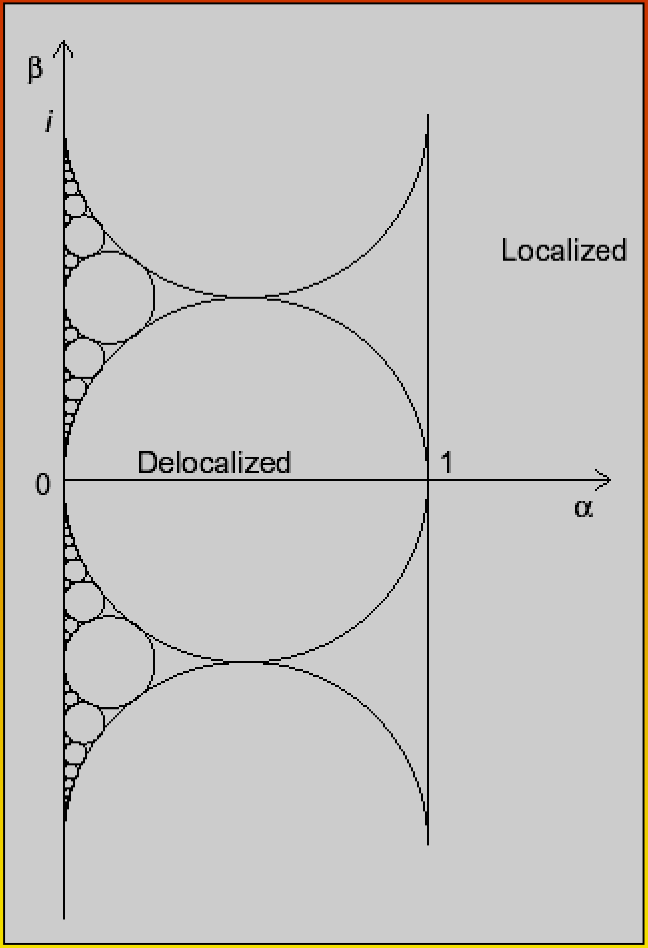
PHASE DIAGRAM ?



Mapping of the line $\alpha=1$ under $z \rightarrow 1/(1 + inz)$

Arguments leading to this phase diagram based mainly on duality, & assumption of localisation for strong coupling to bosonic bath.

The duality is now that of the generalised vector Coulomb gas, in the complex z- plane.



Proposed phase diagram (Callan & Freed, 1992)

C. G. Callan and L. Thorlacius, Nucl. Phys. B 329, 117 (1990).

C. G. Callan and D. Freed, Nucl. Phys. B 374, 543 (1992).

DIRECT CALCULATION of μ (Chen & Stamp)

We add a finite external field:

$$\hat{H}_p = \frac{1}{2m}\left[\hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}}\right]^2 - e\mathbf{E}\cdot\mathbf{r} + U(\hat{\mathbf{r}}) + \sum_\alpha P_\alpha^2/2m_\alpha$$

$$+\frac{1}{2}\sum_\alpha m_\alpha\omega_\alpha^2(x_\alpha + q\lambda_\alpha/m_\alpha\omega_\alpha^2)^2$$

We wish to calculate directly the time evolution of the reduced density matrix of the particle. It is convenient to write this in Wigner form:

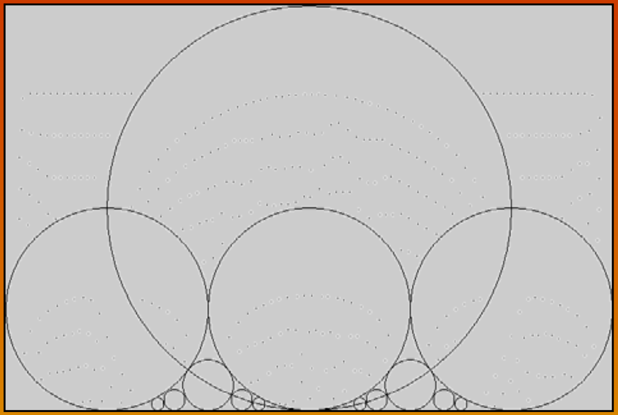
$$\rho(\mathbf{Q},\mathbf{r},t) =< \mathbf{Q} + \frac{\mathbf{r}}{2}|\hat{\rho}(t)|\mathbf{Q} - \frac{\mathbf{r}}{2} >$$

$$W(\mathbf{Q},\mathbf{P},t) = \int_{-\infty}^{\infty} \frac{d^2r}{2\pi\hbar}\rho(\mathbf{Q},\mathbf{r},t)\exp[-\frac{i}{\hbar}\mathbf{P}\cdot\mathbf{r}].$$

$$\rho(\mathbf{Q}_f,\mathbf{r}_f,t) = \int_{-\infty}^{\infty} d^2Q_i d^2r_i J(\mathbf{Q}_f,\mathbf{r}_f,t;\mathbf{Q}_i,\mathbf{r}_i,0) \times \rho(\mathbf{Q}_i,\mathbf{r}_i,0)$$

RESULTS of DIRECT CALCULATION

We get exact results on a particular circle in the phase plane- the one for which K = 1/2



$$\kappa = \frac{2\pi\eta}{k_0^2\hbar}\ \left(= \frac{\eta a_0^2}{2\pi\hbar}\right), \qquad \phi = \frac{2\pi}{k_0^2\hbar}\left(\frac{e|\mathbf{B}|}{c}\right),$$

$$K = \frac{\kappa}{\kappa^2+\phi^2}, \qquad \Phi = \frac{\phi}{\kappa^2+\phi^2}.$$

The reason is that on this circle, one finds that both the long- and short-range parts of the interaction permit a 'dipole' phase, in which the system form close dipoles, with the dipolar widely separated. This happens nowhere else.

One then may immediately evaluate the dynamics, which is well-defined. If we write this in terms of a mobility we have the simple results shown:

$$\frac{\sigma_{xx}}{\sigma_0} = \frac{\sigma_{yy}}{\sigma_0} = 2\Phi[\Phi + \frac{I(\Phi)}{\pi}],$$

$$\frac{\sigma_{xy}}{\sigma_0} = -\frac{\sigma_{yx}}{\sigma_0} = 2\Phi^2\frac{I(\Phi)}{\pi} - \Phi - \frac{I(\Phi)}{\pi},$$

$$\sigma_0 = \frac{2e^2\pi}{k_0^2\hbar}.$$

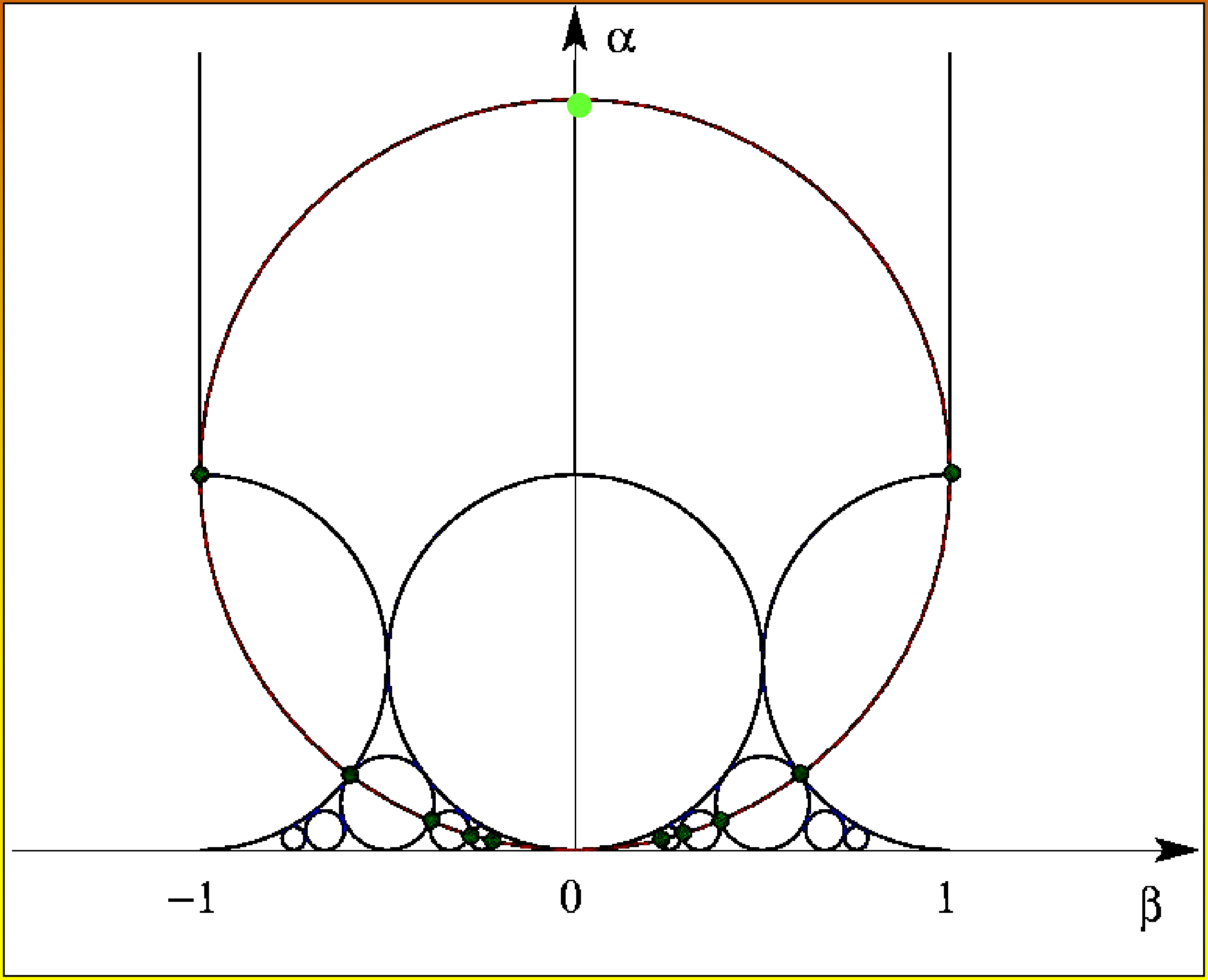
$$I(\Phi) = \tan(2\pi\Phi)\left\{1 - \frac{[1-\cos(2\pi\Phi)]\ln[1+\cos(2\pi\Phi)]}{\cos(2\pi\Phi)}\right\}$$

RESULTS on CIRCLE K = 1/2

The results can be summarized as shown in the figure. For a set of points on the circle the system is localised. At all other points on the circle, it is delocalised.

This behaviour is quite different from the previous results! The explanation is almost certainly the existence of 'hidden fixed points'

The behaviour on this circle should be testable in experiments.



SOME DETAILS of the CALCULATION

In the next 6 pages some details of the calculations are given, for specialists. The dynamics of the density matrix is calculated using path integral methods. We define the propagator for the density matrix as follows:

$$\rho(\mathbf{Q}_f,\mathbf{r}_f,t) = \int_{-\infty}^{\infty} d^2Q_i d^2r_i J(\mathbf{Q}_f,\mathbf{r}_f,t;\mathbf{Q}_i,\mathbf{r}_i,0) \times \rho(\mathbf{Q}_i,\mathbf{r}_i,0)$$

This propagator is written as a path integral along a Keldysh contour:

$$J(\mathbf{Q}_f,\mathbf{r}_f,t;\mathbf{Q}_i,\mathbf{r}_i,0) = \int_{\tau\in\gamma} \mathcal{D}\mathbf{z}(\tau) \exp\left\{\frac{i}{\hbar}S[\mathbf{z}(\tau)]\right\} \times F[\mathbf{z}(\tau)]$$

All effects of the bath are contained in Feynman's influence functional, which averages over the bath dynamics, entangled with that of the particle:

$$F[\mathbf{Q},\mathbf{r}] = \exp\left\{-\frac{1}{\hbar}\int_0^t d\tau \int_0^\tau d\tau' \mathbf{r}(\tau)\cdot\mathbf{r}(\tau')\alpha_2(\tau-\tau')\right.$$

$$\left. -\frac{i}{\hbar}\int_0^t d\tau \left[\int_0^\tau d\tau' \mathbf{r}(\tau)\cdot\dot{\mathbf{Q}}(\tau')\alpha_1(\tau-\tau') + \mathbf{r}(\tau)\cdot\mathbf{Q}_i\alpha_1(\tau)\right]\right\},$$

The 'reactive' part & the 'decoherence' part of the influence functional depend on the spectral function:

$$\alpha_1(\tau) = 2\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{J(\omega)}{\omega}\cos(\omega\tau),$$

$$\alpha_2(\tau) = \hbar\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} J(\omega)\coth\left[\frac{\beta\omega\hbar}{2}\right]\cos(\omega\tau),$$

Influence of the periodic potential

We do a weak potential expansion, using the standard trick

$$e^{iV_o \int^t d\tau \cos x(\tau)} = \sum_{n=0}^{\infty} (iV_o/2)^n \int^t dt_n .. \int^{t_2} dt_1 \sum_{e_j=\pm\}} e^{\int^t dt' \rho(t') x(t')}$$

Without the lattice potential, the path integral contains paths obeying the simple Q Langevin eqtn:

$$m \frac{d^2 \tilde{q}}{dt^2} + \tilde{\eta} \frac{d\tilde{q}}{dt} = E + \tilde{\xi(t)} - \eta \delta(t) \tilde{q}(t=0)$$

The potential then adds a set of 'delta-fn. kicks':

$$m\ddot{Q} + \tilde{\eta}\dot{Q} = -\eta Q_i \delta(t) + \xi(t) + eE + \frac{k_0 \hbar}{2} \left[\sum_{j=1}^{n_1} \sigma_{1j} \delta(t - t_{1j}) + i \sum_{j=1}^{n_2} \sigma_{2j} \delta(t - t_{2j}) \right]$$

One can calculate the dynamics now in a quite direct way, not by calculating an autocorrelation function but rather by evaluating the long-time behaviour of the density matrix.

If one evaluates the long-time behaviour of the Wigner function one then finds the following, after expanding again in the potential (cf. Schmid problem):

$$v_E = \lim_{t\to\infty} \left\langle \frac{\hat{x} + i\hat{y}}{t} \right\rangle (t) = \lim_{t\to\infty} \left[\int_{-\infty}^{\infty} d^2Q_f d^2P_f \frac{W(Q_f, P_f, t)}{t} \right]$$

$$= \frac{eE}{\tilde{\eta}} + \lim_{t\to\infty} \frac{1}{t} \sum_{n_1+n_2\neq 0} \left[\frac{V_0}{\hbar}\right]^{n_1+n_2} \sum_{\{\sigma_{\alpha j}=\pm 1\}} \int_0^t \mathcal{D}\{t_{1j}\} \int_0^t \mathcal{D}\{t_{2j}\}$$

$$\times \left\langle \frac{k_0\hbar}{2} [\sigma_{1n_1} g(t - t_{1n_1}) + i\sigma_{2n_2} g(t - t_{2n_2})] \prod_{\alpha=1}^{2} \prod_{j=1}^{n_\alpha} \{\sigma_{\alpha j} \sin[k_0 Q_\alpha(t_{\alpha j})]\} \right\rangle$$

$$= \frac{eE}{\tilde{\eta}} + \sum_{\alpha=1}^{2} v_1^{(\alpha)}(E) + v_2(E).$$

We now go to some rather detailed exact results for this velocity, in the next 3 slides ….

LONGITUDINAL COMPONENT:

$$v_1^{(1)}(E) = \frac{k_0\hbar}{2\tilde{\eta}} \sum_{n>0}^{\infty} \sum_{\{\mu_j=\pm 1\}} \left[\frac{V_0}{\hbar}\right]^n (-1)^{n/2}(-i) \int_{-\infty}^{0} \mathcal{D}\{t_j\} F_1 \times F_2 \Big|_{\sum_j \mu_j = 0} ;$$

$$F_1 = \prod_{k=1}^{n-1} \sin\left\{\frac{k_0^2\hbar}{2} \sum_{j=k+1}^{n} \mu_j g_1(t_j - t_k)\right\},$$

$$F_2 = \exp\left\{ik_0 \sum_{j=1}^{n} \tilde{E}_1 \mu_j t_j + \frac{k_0^2}{2} \sum_{j,j'=1}^{n} \mu_j \mu_{j'} C_1(t_j - t_{j'})\right\}.$$

$$\begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{pmatrix} = \begin{pmatrix} e/\eta_1 & e/\eta_2 \\ -e/\eta_2 & e/\eta_1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$\eta_1 = [(e|\mathbf{B}|/c)^2 + \eta^2]/\eta,$$

$$\eta_2 = [(e|\mathbf{B}|/c)^2 + \eta^2]/(e|\mathbf{B}|/c)$$

TRANSVERSE COMPONENT:

$$v_2(E) = \sum_{n_1,n_2>0}^{\infty} \left[\frac{V_0}{\hbar}\right]^{n_1+n_2} (-1)^{(n_1+n_2)/2} \int_{-\infty}^{0+} \mathcal{D}\{t_{1j}\} \int_{-\infty}^{+0} \mathcal{D}\{t_{2j}\}$$

$$\times (-i)\frac{k_0\hbar}{2\tilde{\eta}} \sum_{\{\mu_{\alpha j}=\pm 1\}} G_1 \times G_2 \times G_3 \Big|_{\sum_j \mu_{\alpha j}=0} ,$$

$$G_1 = \delta(t_{1n_1}) \sin\left[-\frac{k_0^2\hbar}{2}\sum_{j=1}^{n_1} \mu_{1j} g_2(t_{1j} - t_{2n_2})\right] + i\delta(t_{2n_2}) \sin\left[\frac{k_0^2\hbar}{2}\sum_{j=1}^{n_2} \mu_{2j} g_2(t_{2j} - t_{1n_1}),\right]$$

$$G_2 = \prod_{\alpha=1}^{2} \prod_{k=1}^{n_\alpha - 1} \sin\left\{\frac{k_0^2\hbar}{2}\left[\sum_{j=k+1}^{n_\alpha} \mu_{\alpha j} g_1(t_{\alpha j} - t_{\alpha k})\right.\right.$$

$$\left.\left. + \sum_{\alpha'=1}^{2} \epsilon_{\alpha\alpha'} \sum_{j=1}^{n_{\alpha'}} \mu_{\alpha' j} g_2(t_{\alpha' j} - t_{\alpha k})\right]\right\}$$

$$G_3 = \exp\left\{\sum_{\alpha=1}^{2}\left[ik_0 \sum_{j=1}^{n_\alpha} \mu_{\alpha j} t_{\alpha j} \tilde{E}_\alpha + \frac{k_0^2}{2} \sum_{j,j'=1}^{n_\alpha} \mu_{\alpha j}\mu_{\alpha j'} C_1(t_{\alpha j} - t_{\alpha j'})\right]\right.$$

$$\left. + k_0^2 \sum_{j=1}^{n_1} \sum_{j'=1}^{n_2} \mu_{1j}\mu_{2j'} C_2(t_{1j} - t_{2j'})\right\}$$

DIAGONAL & CROSS-CORRELATORS:

$$C_1(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \eta\omega\hbar \coth[\frac{\beta\hbar\omega}{2}][1 - \cos(\omega t)]$$

$$\times \frac{[(m\omega)^2 + \eta^2 + (e|\mathbf{B}|/c)^2]}{\omega^2 \{[(m\omega)^2 - \eta^2 - (e|\mathbf{B}|/c)^2]^2 + 4(m\omega\eta)^2\}},$$

$$C_2(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \eta\omega\hbar \coth[\frac{\beta\hbar\omega}{2}] \sin(\omega t)$$

$$\times \frac{2[(m\omega)(e|\mathbf{B}|/c)]}{\omega^2 \{[(m\omega)^2 - \eta^2 - (e|\mathbf{B}|/c)^2]^2 + 4(m\omega\eta)^2\}}.$$

It turns out from these exact results that not all of the conclusions which come from a simple analysis of the long-time scaling are confirmed. In particular we do not get the same phase diagram as Callan et al., but instead the results that were summarized a few pages back, for the circle K = ½.

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SUMMARY of COURSE

1. We have seen how in the quest for large-scale quantum phenomena in magnetic systems, one ends up looking at phenomena very different from those occurring in superfluids or superconductors. In particular one has to look at models of coupled 2-level systems, often dipolar-coupled; and some of the really interesting effects come form magnetic solitons. Topology and topological phase in spin space play an important role, for solitons, interacting spin qubits, and in decoherence. Decoherence enters in a very sophisticated way, and one needs to employ rather new models for decoherence, and to describe low-T quantum environments. Issues of principle can be very important: in particular one has to deal with environments that are far from equilibrium, where linear response and fluctuation-dissipation theorems are rather meaningless.

2. In applications to experiment the question of what sort of environment one is dealing with becomes terribly important. The 'spin bath', describing localised modes, plays a central role because most of the important low-energy environmental modes are of this type. In dealing with tunneling phenomena one finds that without this spin bath, very little would be seen – it is the nuclear spins that 'liberate' tunneling, which would otherwise be blocked. On the other hand most coherence phenomena are rapidly destroyed by the spin bath, even though it causes little or no dissipation. A whole variety of large-scale quantum phenomena involving tunneling spins and tunneling topological solitons have been seen, and agreement with theoretical predictions is pretty good.

3. New models in the field are considerable interest. These include models of 'quantum walks', as well as models describing topological quantum fluids. These models are of very general interest, in magnetism and elsewhere

THANKS TO ALL PARTICIPANTS, and HAVE A GOOD SUMMER !!

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